2011 Team Essay Solutions

1. 7(4) = 28, 7(3) = 21, 7(3) + 8 = 29, 7(2) + 8 = 22, 7(2) + 8(2) = 30, 7 + 8(2) = 23, 8(3) = 24. Then 21, 22, 23, 24, 28, 29, and 30 are the integers from 21 through 30 attainable from 7 and 8.

2.

32	37	42	47	52	57	62	67
24	29	34	39	44	49	54	59
16	21	26	31	36	41	46	51
8	13	18	23	28	33	38	43
0	5	10	15	20	25	30	35

3.

32	37	2	7	12	17	22	27
24	29	34	39	4	9	14	19
16	21	26	31	36	1	6	11
8	13	18	23	28	33	38	3
0	5	10	15	20	25	30	35

4. 1, 2, 3, 4, 6, 7, 9, 11, 12, 14, 17, 19, 22, 27.

5. If c is attainable from a and b, we can write c = as + bt for nonnegative integers s and t. Writing

$$c + ab = as + bt + ab = a(s + b) + bt$$

shows that c + ab = au + bt for the nonnegative integers u = s + b and t. Thus, c + ab is attainable from a and b.

6. Let n be an integer from ab through 2ab – 1. Setting m = n – ab gives an integer m from 0 through ab – 1. The reduced array for a and b contains m, by Claim 1.

If m is below or to the left of the boundary path in the reduced array, then it also appears in the first array. This means that m = ax + by for integers x from 0 through b – 1 and y from 0 through a – 1. Then m is attainable from a and b, and so is n = m + ab, by Problem 5.

On the other hand, suppose that m is above or to the right of the boundary path in the reduced array. Then n = m + ab is in the first array. This means that n = ax + by for integers x from 0 through b – 1 and y from 0 through a – 1, and so n is attainable from a and b.

We have shown that all integers from ab through 2ab – 1 are attainable from a and b. Adding ab shows that all integers from 2ab through 3ab – 1 are attainable from a and b, by Problem 5. Adding ab again shows that all integers from 3ab through 4ab – 1 are attainable from a and b, by Problem 5. Continuing in this way shows that all integers greater than or equal to ab are attainable from a and b.

7. The second paragraph of the answer to Problem 6 shows that the entries of the reduced array below or to the left of the boundary path are all attainable from a and b. Conversely, let m be an entry in the reduced array that is attainable from a and b. Since m is attainable, we have m = ax + by for nonnegative integers x and y. Since m is in the reduced array, it is less than ab (by Claim 1), and so $x \le b - 1$ and $y \le a - 1$. Then m appears in the first array for a and b, and the fact that m is less than ab means that m is below or to the left of the boundary path in both arrays.

Any integer greater than or equal to ab is attainable from a and b (by Problem 6) and does not appear in the reduced array for a and b (by Claim 1). Any nonnegative integer less than ab appears exactly once in the reduced array for a and b (by Claim 1), and it is below or to the left of the boundary path if and only if it is attainable from a and b (by the previous paragraph). Thus, the nonnegative integers that are unattainable from a and b are exactly the entries of the reduced array above or to the right of the boundary path.

8. Suppose that

$$ax + by = au + bv. \tag{3}$$

Collecting terms gives

$$a(x - u) = b(v - y). \tag{4}$$

If $x \neq u$, we can divide both sides of (4) by $x - u$ and b (since $b > 0$) and get

$$\frac{a}{b} = \frac{v - y}{x - u}.$$

By Fact 1, we have

$$x - u = bt \tag{5}$$

for an integer t. Since x and u are integers from 0 through $b - 1$, we have

$$x - u \leq x < b$$

and

$$x - u \geq -u > -b.$$

Thus, we must have $t = 0$ in (5). This contradicts the assumption that $x \neq u$ and shows that (3) cannot hold.

9. Suppose that

$$ax + by = au + bv - ab. \tag{6}$$

Collecting terms gives

$$a(x - u + b) = b(v - y). \tag{7}$$

Since x and u are integers from 0 through $b - 1$, we have

$$x - u + b \geq -u + b > 0 \tag{8}$$

and

$$x - u + b \leq x + b < 2b. \tag{9}$$

We can divide (7) by $x - u + b$ (since this is nonzero, by (8)) and by b (since $b > 0$) to get

$$\frac{a}{b} = \frac{v - y}{x - u + b} .$$

Fact 1 gives $v - y = at$ and $x - u + b = bt$. By (8) and (9), we must have $x - u + b = b$, $t = 1$, and $v - y = a$. This is impossible, since

$$v - y \leq v < a$$

(because v and y are integers from 0 through $a - 1$). Thus, (6) cannot hold.

10. The entries in the first array are $ax + by$ for integers x from 0 through $b - 1$ and y from 0 through $a - 1$. By Problem 8, no two entries in different columns of the first array are equal. No two entries in the same column of the first array are equal, since

each entry is b more than the one below it. Thus, no two entries of the first array are equal.

It follows that no two entries of the reduced array are equal when both lie below or to the left of the boundary path (since they equal entries in the same positions of the first array) or both lie above or to the right of the boundary path (since they are each ab less than the entries in the same positions of the first array). By Problem 9, no two entries of the reduced array are equal when one lies below or to the left of the boundary path and one lies above or to the right. The last two sentences show that no two entries of the reduced array are equal.

11. Let k be an entry of the reduced array below or to the left of the boundary path. Since k is also in the first array, it equals ax + by for nonnegative integers x and y, and so it is nonnegative. It is also less than ab, by the choice of the boundary path. In short, we have $0 \le k < ab$.

Let m be an entry of the reduced array above or to the right of the boundary path. We have

$$m = ax + by - ab$$

for $0 \le x \le b - 1$ and $0 \le y \le a - 1$. It follows that

$$m \le a(b - 1) + b(a - 1) - ab = ab - a - b < ab.$$

We also have $m \ge 0$, since the corresponding entry ax + by in the first array is greater than or equal to ab, by choice of the boundary path. In short, we have $0 \le m < ab$.

We have shown that every entry of the reduced array is an integer from 0 through ab – 1. There are ab such integers. There are also ab entries in the reduced array (since it has a rows and b columns), and no two of these entries are equal (by Problem 10). Thus, the reduced array contains each integer from 0 through ab – 1 exactly once and has no other entries.