MATH FIELD DAY SOLUTIONS 2012

SOLUTIONS TO 2012
TEAM ESSAY

1.

Step 0.	0:0, 0:0
Step 1.	5:0, 0:5
Step 2.	5:5, 2:8
Step 3.	2:0, 0:2
Step 4.	5:2, 0:7
Step 5.	5:7, 4:8
Step 6.	4:0, 0:4
Step 7.	5:4, 1:8
Step 8.	1:0, 0:1
Step 9.	5:1, 0:6
Step 10.	5:6, 3:8
Step 11.	3:0, 0:3
Step 12.	5:3, 0:8
Step 13.	5:8, 5:8

2. 5, 10, 2, 7, 12, 4, 9, 1, 6, 11, 3, 8, 13

3. (a) Filling the m-jug from the pool h times and emptying the n-jug into the pool k times changes the total number of quarts in the two jugs by $mh - nk$. Because there is no change in the total number of quarts in the jugs from the start of step r to the start of step s (since $y_r = y_s$), we have

$$mh = nk. \tag{4}$$

Since $r < s$, h or k is positive. Since m and n are also positive, (4) shows that h and k are both positive. Dividing both sides of (4) by nh gives

$$\frac{m}{n} = \frac{k}{h}.$$

By Claim 1, we have $k = mt$ and $h = nt$ for a positive integer t.

(b) The number of times we perform part B in steps r through $s - 1$ is

$$(s - 1) - (r - 1) = s - r.$$

Since these times are divided between the h times we fill the m-jug from the pool and the k times we empty the n-jug into the pool, we have

$$s - r = h + k.$$

Substituting $h = nt$ and $k = mt$ from (a) gives

$$s - r = nt + mt = (n + m)t,$$

and this is at least $m + n$ (since $t \geq 1$, by (a)).

4. Since we start with both jugs empty, we have $y_0 = 0$. For any positive integer j, we have

$$y_j = mh - nk,$$

where h is the number of times we fill the m-jug from the start of step 0 to the start of step j and k is the number of times we empty the n-jug. Thus, y_j is an integer.

The total number of quarts in the two jugs cannot exceed their combined capacity. Thus, we have $y_j \leq m + n$ for each j.

Since both jugs are initially empty, we fill the m-jug from the pool in part B of step 0, giving $y_1 = m$. In performing part B of any step, we don't empty the n-jug into the pool if the m-jug is empty. Thus, there is water in at least one jug from step 1 on, and y_j is positive for each positive integer j. Since y_j is an integer, it is at least 1.

5. Let r and s be two integers from 1 through $m+n$ with $r < s$. Since $s \leq m+n$ and $r \geq 1$, we have

$$s - r \leq m + n - 1 < m + n.$$

Then y_r does not equal y_s by Problem 3(b).

The sequence $y_1 ..., y_{m+n}$ lists integers from 1 through $m + n$, by Problem 4. The sequence (1) contains $m + n$ different integers, by the paragraph above. Since there are only $m + n$ integers from 1 through $m + n$, each of them appears exactly once in (1). Thus, (1) consists of the integers 1 through $m + n$ in some order.

6. (a) Filling the m-jug from the pool h times and emptying the n-jug into the pool k times changes the total number of quarts in the two jugs by $mh - nk$. Because the change from the start of step 0 to the start of step s is $m + n$ (since $y_0 = 0$ and $y_s = m + n$), we have

$$mh - nk = m + n.$$

Collecting the multiples of m on the left and the multiples of n on the right gives

$$m(h - 1) = n(k + 1). \tag{5}$$

Since m, n, and $k + 1$ are all positive, so is $h - 1$ (by (5)). Dividing both sides of (5) by $n(h - 1)$ gives

$$\frac{m}{n} = \frac{k + 1}{h - 1}.$$

By Claim 1, we have $k + 1 = mt$ and $h - 1 = nt$ for a positive integer t. Rewriting these equations gives $h = nt + 1$ and $k = mt - 1$.

(b) The number of times we perform part B in steps 0 through $s-1$ is s (since there are $s-1$ steps from 1 through $s-1$ plus one for step 0). Since these times are divided between the h times we fill the m-jug from the pool and the k times we empty the n-jug into the pool, we have

$$s = h + k.$$

Substituting $h = nt + 1$ and $k = mt - 1$ from (a) gives

$$s = nt + 1 + mt - 1 = (m + n)t,$$

and this is at least $m + n$ (since $t \geq 1$, by (a)).

7. (a) By Problem 5, one of the integers $y_1, \ldots, y_{m+n}$ is $m + n$. None of the integers $y_1, \ldots, y_{m+n-1}$ is $m + n$, by Problem 6(b). Thus, y_{m+n} is $m + n$.

(b) By (a), we have $y_{m+n} = m + n$, and both jugs are full in step $m + n$. Then, we empty the n-jug into the pool in part B of step $m + n$ and get $y_{m+n+1} = m$. Since both jugs are empty in step 0, we fill the m-jug from the pool in part B of step 0 and get $y_1 = m$. This shows that $y_{m+n+1} = y_1$.

For each positive integer j, y_j determines the value of y_{j+1}. In fact, if $y_j \leq n$, the m-jug is empty after part A of step j, and we fill it in part B to get $y_{j+1} = y_j + m$. If $y_j >$ n, the m-jug is not empty after part A of step j and the n-jug is full, and we empty the n-jug into the pool in part B of step j to get $y_{j+1} = y_j - n$.

We have $y_{m+n+1} = y_1$ (by the second-to-last paragraph). Since y_{m+n+1} determines y_{m+n+2} in the same way as y_1 determines y_2 (by the previous paragraph), we have $y_{m+n+2} = y_2$. Since y_{m+n+2} determines y_{m+n+3} in the same way as y_2 determines y_3, we have $y_{m+n+3} = y_3$. Continuing in this way shows that $y_{m+n+j} = y_j$ for all positive integers j.

8. Multiplying both sides of the equation $\frac{a}{b} = \frac{c}{d}$ by bd gives

$$ad = bc. \tag{6}$$

(a) If $b = dt$, substituting in (6) gives

$$ad = dtc.$$

Dividing both sides by d (since d $\neq 0$) gives $a = ct$.

(b) Equation 6 implies that

$$ab - adt = ab - bct.$$

Factoring a out the left side and b out of the right gives

$$a(b - dt) = b(a - ct). \tag{7}$$

If $b - dt > 0$, (7) shows that $a - ct > 0$ (since a and b are positive). Dividing both sides of (7) by $b(b - dt)$ gives

$$\frac{a}{b} = \frac{a - ct}{b - dt}.$$

9. Let t be the largest integer such that $b - dt \geq 0$. By the paragraph before Problem 9, we have

$$0 \leq b - dt < d.$$

If we had $b - dt > 0$, then we would have $a - ct > 0$ and

$$\frac{a - ct}{b - dt} = \frac{a}{b}$$

by Problem 8(b). Since $b - dt < d$, this would contradict the choice of c and d as positive integers such that $\frac{c}{d} = \frac{a}{b}$ and d is as small as possible. This contradiction shows that $b = dt$ and $a = ct$, by Problem 8(a) (since $b - dt \geq 0$). Since b and d are positive, so is t.

10. Taking a and b in Problem 9 to be m and n shows that $m = ct$ and $n = dt$ for a positive integer t. Since m and n have no common factor greater that 1, we have $t = 1$, $m = c$, and $n = d$.