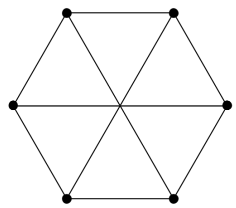
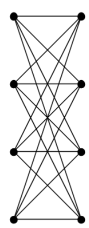
MATH FIELD DAY 2013

Problem 1. The hexagon in Figure 5 is a triangle-free graph with 6 vertices and 6 edges. Show how to add 3 more edges and still have a triangle-free graph. No explanation is needed.



Solution. □

Problem 2. Draw T_8. How many edges does it have?

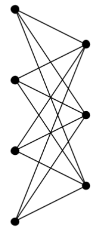


16 edges

Solution. □

Problem 3. Use Mantel's Theorem to draw the triangle-free graph with 7 vertices that has the greatest possible number of edges. How many edges does it have?

Solution. □



Problem 4. Deduce from Mantel's Theorem that the greatest possible number of edges in a triangle-free graph with n vertices is $n^2/4$ if n is even and $(n^2-1)/4$ if n is odd.

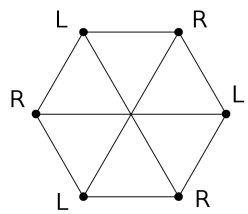
Solution. If n is even, L and R have $n/2$ vertices apiece. Joining each vertex in L with each vertex in T gives $(n/2)^2 = n^2/4$ edges.

If n is odd, L has $(n+1)/2$ vertices and R has $(n-1)/2$. Joining each vertex in L with each vertex in R gives

$$\left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right) = \frac{n^2-1}{4}$$

edges. □

Problem 5. Draw a copy of your answer to Problem 1. Label three of the vertices L and the other three vertices R so that the edges of the graph are all the segments with one vertex labeled L and labeled R. (This shows that the graph is T_6.)

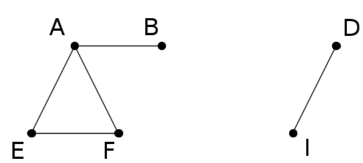


Solution. □

Problem 6. Show that T_n is triangle-free for every positive integer n.

Solution. Any set of three vertices of T_n must include two vertices from L or two from R. These two vertices are not the endpoints of an edge of T_n. □

Problem 7. For the graph G in Figure 10, draw $G - \{C, H\}$.



Solution. □

Problem 8. Let AB be an edge in T_{n+2} for a positive integer n. Show that $T_{n+2} - \{A, B\}$ is T_n and that T_{n+2} has $n + 1$ more edges than T_n.

Solution. Let L and R be the two sets of vertices forming T_{n+2}. Since AB is an edge of T_{n+2}, one endpoint is in L and the other is in R. Switching A and B, if necessary, let us suppose that A is in L and B is in R. Let L' be the set of vertices in L other than A, and let R' be the set of vertices other than B.

Because L and R divide up the $n+2$ vertices of T_{n+2}, L' and R' divide up the n vertices of $T_{n+2} - \{A, B\}$. Because L has either the same number of vertices as R or one more, L' has either the same number of vertices as R' or one more. Because the edges of T_{n+2} are the segments with one endpoint in L and one in R, the edges of $T_{n+2} - \{A, B\}$ are the segments with one endpoint in L' and one in R'. Thus, $T_{n+2} - \{A, B\}$ is T_n.

In reducing T_{n+2} to T_n, we have removed AB, one edge from A to each vertex in R', and one edge from B to each vertex in L'. Since L' and R' divide up the n vertices of T_n, we have removed a total of $n+1$ edges. □

Problem 9. Let G be a triangle-free graph with $n+2$ vertices for a positive integer n, and let AB be an edge of G. Show that G has at most $n+1$ more edges than $G - \{A, B\}$.

Solution. Let Z be a vertex of G other than A and B. Since G has edge AB and contains no triangles, it has at most one of the edges AZ and BZ. Thus, since G has n vertices other than A and B, it has at most n edges having exactly one endpoint equal to A or B. Along with AB, these are the edges we remove from G to get $G - \{A, B\}$. Thus, G has at most $n+1$ more edges than $G - \{A, B\}$. □

Problem 10. Let G be a triangle-free graph with $n+2$ vertices for a positive integer n, and let AB be an edge of G. If $G - \{A, B\}$ is T_n and G has $n+1$ more edges than T_n, prove that G is T_{n+2}.

Solution. Since $G - \{A, B\}$ is T_n, its vertices are divided into two sets L' and R' such that the edges of $G - \{A, B\}$ are all segments with one endpoint in L' and one in R', where L' has either the same number of vertices as R' or one more. Since G has $n+1$ more edges than $G - \{A, B\}$, the solution to Problem 9 shows that the edges of G not in $G - \{A, B\}$ are AB and exactly one of the two edges AZ and BZ for each Z in L' or R'.

For any vertices X in L' and Y in R', G cannot have both edges AX and AY, and it cannot have both edges BX and BY, since G has edge XY and contains neither triangle AXY or BXY. Thus, after possibly switching A and B, the last sentence of the previous paragraph shows that the edges of G not in $G - \{A, B\}$ are AB, AY for all vertices Y in R', and BX for all vertices X in L'.

Form sets L and R of vertices of G by adding A to L' and B to R'. Since L' and R' divide up the vertices of $G - \{A, B\}$, L and R divide up the vertices of G. Since L' has either the same number of vertices as R' or one more, L has either the same number of vertices of R or one more. Since the edges of $G - \{A, B\}$ are the segments with one endpoint in L' and one in R', the last sentence of the previous paragraph shows that the edges of G are the segments with one endpoint in L and one in R. Thus, G is the graph T_{n+2} determined by the sets L and R. □

Problem 11. Verify directly that Statement I is true for $n = 1$ and $n = 2$.

Solution. T_1 consists of one vertex and no edges. Since there are no other graphs with one vertex, Statement I is true for $n = 1$. T_2 consists of two vertices joined by an edge. Since the only other graph with two vertices has no edges, Statement I holds for $n = 2$. □

Problem 12. Let k be a positive integer such that Statement I is true for $n = k$. Use Problems 8-10 to show that Statement I is true for $n = k + 2$.

Solution. Let $e(H)$ denote the number of edges in any graph H. Let G be a triangle-free graph with $k + 2$ vertices. If G has no edges, it has fewer edges than T_{k+2} (since $k + 2 \geq 3$). Thus, we can assume that G has an edge AB.

Since G is a triangle-free graph with $k + 2$ vertices, $G - \{A, B\}$ is a triangle-free graph with k vertices. Because Statement I holds for $n = k$, we have

$$e(G - \{A, B\}) \leq e(T_k). \tag{1}$$

We also have

$$e(G) - e(G - \{A, B\}) \leq k + 1, \tag{2}$$

by Problem 9. The left sides of (1) and (2) sum to $e(G)$, and the right sides sum to $e(T_{k+2})$ (by Problem 8). Thus, we have

$$e(G) < e(T_{k+2})$$

unless equality holds in both (1) and (2). Equality in (1) implies that $G - \{A, B\}$ is T_k (by Statement I for $n = k$), and then equality in (2) implies that G is T_{k+2} (by Problem 10).

We have proved that every triangle-free graph with $k+2$ vertices that doesn't equal T_{k+2} has fewer edges than T_{k+2}. Thus, Statement I holds for $n = k + 2$. □

Problem 13. Explain why Mantel's Theorem follows from Problems 6, 11, and 12.

Solution. By Problem 11, Statement I holds for $n = 1$ and $n = 2$. Then Problem 12 shows that Statement I holds for $n = 3$ and $n = 4$. Using Problem 12 again show that Statement I holds for $n = 5$ and $n = 6$. Continuing to apply Problem 12 in this way shows that Statement I holds for all positive integers.

For every positive integer n, T_n is a triangle-free graph with n vertices (by Problem 6), and it has more edges than every other triangle-free graph with n vertices (by the previous paragraph). Thus, Mantel's Theorem holds. □