2016 MATH FIELD DAY
HUDDLE PROBLEMS AND SOLUTIONS

Problem 1. Evaluate

$$\frac{1}{\sqrt{1}+\sqrt{0}}+\frac{1}{\sqrt{2}+\sqrt{1}}+\frac{1}{\sqrt{3}+\sqrt{2}}+\cdots+\frac{1}{\sqrt{2015}+\sqrt{2014}}$$

Proof. For each k, we have

$$\frac{1}{\sqrt{k+1}+\sqrt{k}}=\frac{1}{\sqrt{k+1}+\sqrt{k}}\cdot\frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k+1}-\sqrt{k}}=\sqrt{k+1}-\sqrt{k}.$$

Applying this to each summand, the sum in question is replaced by

$$(\sqrt{1}-\sqrt{0})+(\sqrt{2}-\sqrt{1})+(\sqrt{3}-\sqrt{2})+(\sqrt{4}-\sqrt{3})+\cdots+(\sqrt{2015}-\sqrt{2014})=\boxed{\sqrt{2015}.}$$

□

Problem 2. The *oblocity* of a right circular cylinder (open on both ends) is defined to be its volume (in cubic inches) minus the area of its curved surface (in square inches). When the cylinder's radius is tripled, its oblocity is multiplied by 10. What is the radius of the cylinder, in inches?

Solution. Suppose the cylinder has radius r and height h. Then its volume is $\pi r^2 h$ and its surface area is $2\pi rh$, giving a formula for the oblocity as $\pi r^2 h - 2\pi rh$. We are given the following ratio of oblocities:

$$10=\frac{\pi(3r)^2h-2\pi(3r)h}{\pi r^2h-2\pi rh}=\frac{3(3r-2)}{r-2}.$$

Clearing denominators gives $10r-20=9r-6$, giving $\boxed{r=14.}$ □

Problem 3. For any real number r, let $\lfloor r\rfloor$ be the greatest integer less than or equal to r. For example, we have $\lfloor 2\frac{3}{4}\rfloor=2$ and $\lfloor 3\rfloor=3$. In effect, $\lfloor r\rfloor$ rounds r down to an integer. Set

$$f(n)=\left\lfloor\frac{n}{1}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor+\cdots+\left\lfloor\frac{n}{n}\right\rfloor$$

for each positive integer n. Evaluate $f(100)-f(99)$.

Solution. By pairing off terms with the same denominator, we have

$$f(100)-f(99)=\left(\lfloor\tfrac{100}{1}\rfloor-\lfloor\tfrac{99}{1}\rfloor\right)+\left(\lfloor\tfrac{100}{2}\rfloor-\lfloor\tfrac{99}{2}\rfloor\right)+\left(\lfloor\tfrac{100}{3}\rfloor-\lfloor\tfrac{99}{3}\rfloor\right)+\cdots+\left(\lfloor\tfrac{100}{99}\rfloor-\lfloor\tfrac{99}{99}\rfloor\right)+\lfloor\tfrac{100}{100}\rfloor.$$

Now each of these differences $\left\lfloor\frac{100}{k}\right\rfloor-\left\lfloor\frac{99}{k}\right\rfloor$ evaluates to 1 if 100 is a multiple of k (for then k goes into 100 one more time than it goes to 99), and 0 otherwise. That is, the above sum differences contains one $+1$ for each denominator k which divides evenly into 100. Since there are 9 such terms ($k=1,2,4,5,10,20,25,50,100$), the sum evaluates to $\boxed{9.}$ □

Problem 4. Find the smallest positive angle θ, in radians, for which

$$\ln(\sin(3\theta))=\frac{\ln(\cos(6\theta))}{2}+\ln\left(\frac{1}{\sqrt{2}}\right).$$

Solution. One of many possible derivations: Write $\ln\left(\frac{1}{\sqrt{2}}\right)$ as $\frac{1}{2}\ln\left(\frac{1}{2}\right)$, and multiply both sides by 2 to obtain $2\ln(\sin(3\theta))=\ln(\cos(6\theta))+\ln(1/2)$. Combining a log identity and the identity $\sin^2(x)=\frac{1}{2}(1-\cos(2x))$, we re-write the left-hand side:

$$2\ln(\sin(3\theta))=\ln(\sin^2(3\theta))=\ln(\tfrac{1}{2}(1-\cos(6\theta)).$$

Similarly, the right-hand side is $\ln\left(\frac{1}{2}\cos(6\theta)\right)$, so equating them and exponentiating gives

$$\frac{1}{2}(1-\cos(6\theta))=\frac{1}{2}\cos(6\theta).$$

We conclude that $\cos(6\theta)=\frac{1}{2}$. The least positive solution has $6\theta=\frac{\pi}{3}$, so $\boxed{\theta=\frac{\pi}{18}.}$ □