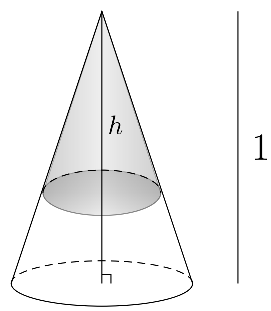
2016 MATH FIELD DAY
LEAP FROG PROBLEMS

Problem 1. Below is a right circular cone of height 1, divided in two parts (the shaded part and the unshaded part) by a disk h units down from the vertex. Find h if the lateral surface area of the shaded region is twice that of the unshaded region.



The "lateral surface area" of such a region is the area of its curved sloping surface. It does not include the area of any horizontal disks.

Problem 2. A quadratic polynomial

$$f(x) = ax^2 + bx + c$$

has positive integer coefficients $a < b < c$ and satisfies:

- $f(0) = 49$ and $f(-7) = 210$
- $f(b) - f(a)$ is prime.

Find $f(1)$.

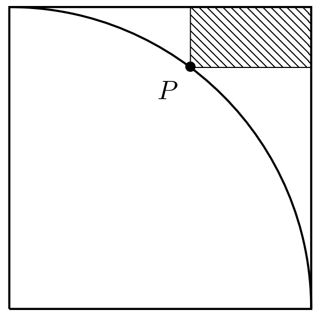
Problem 3. A 2-player game begins with a stack of n coins. Players alternate taking coins from the stack, with the restriction that they can only take a prime number of coins each turn. The first player who can't make a legal move (that is, there are 0 or 1 coins remaining) loses. For example, if $n = 6$, then the first player wins immediately by taking 5 coins, after which the second player has no legal moves. The smallest integer $n \geq 2$ for which the *second* player can guarantee a victory with best play is $n = 9$, and the second smallest is $n = 10$. What is the third smallest?

Problem 4. Twelve cards, numbered from 1-12, are distributed, three each, to four logicians (who cannot see each others' cards). They make the statements below in order. Each statement is true and logically deducible by that logician from their cards and the previous statements.

Johnny:	At least two of my cards are prime.
Ben:	The product of my cards is even.
Reed:	The product of two my cards is the third. Also, Sue and I don't have the same number of cards with even numbers on them.
Sue:	I have a card that is numbered 2 more than Reed's smallest card.

What is the sum of Reed's cards?

Problem 5. The figure below shows a quarter-circle inscribed in a square. A point P is drawn on the circle, and a shaded rectangle with edges parallel to those of the square.



If the shaded rectangle has area 25 and perimeter 52, find the radius of the circle.

Problem 6. Suppose $f(x)$ is a function such that

$$f(x) = 3f\left(\frac{1}{1-x}\right) - 1$$

for all real numbers x except 0 and 1. Find $f(2016)$.

Problem 7. Evaluate

$$\sqrt{142857 \times 90909 \times 77}$$

Problem 8. Below, each of the letters appearing stands for a different numeral from 0 to 9, in such a way that the resulting addition is true.

$$\begin{array}{ccccc} & M & A & T & H \\ + & & D & A & Y \\ \hline F & I & E & L & D \end{array}$$

For instance, MATH is a 4-digit integer with thousands digit M, hundreds digit A, tens digit T, and ones digit H. Note that no positive integer has 0 as its left-most digit.

What 4-digit integer is TIME?