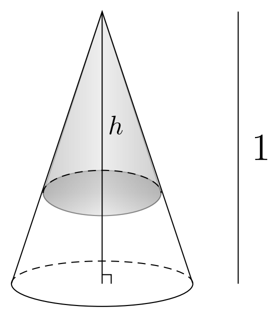
2016 MATH FIELD DAY
LEAP FROG PROBLEMS AND SOLUTIONS

Problem 1. Below is a right circular cone of height 1, divided in two parts (the shaded part and the unshaded part) by a disk h units down from the vertex. Find h if the lateral surface area of the shaded region is twice that of the unshaded region.



The "lateral surface area" of such a region is the area of its curved sloping surface. It does not include the area of any horizontal disks.

Solution. The area of a right circular cone of radius r and slant height s is given by πrs, as can be seen by unraveling the cone into the sector of the circle with radius s which has arc length $2\pi r$. This region has area $\pi s^2 \cdot \frac{2\pi r}{2\pi s} = \pi rs$. Suppose the full cone has radius r and slant height s, so that by similarity, the shaded cone has radius hr and slant height hs. The area of the shaded cone is $\pi(hr)(hs)$, and the area of the unshaded region is $\pi rs - \pi(hr)(hs)$. Setting

$$\pi(hr)(hs) = 2\left(\pi rs - \pi(hr)(hs)\right)$$

we cancel πrs from both sides to get $h^2 = 2(1-h^2)$, so $\boxed{h = \sqrt{2/3}}$. □

Problem 2. A quadratic polynomial

$$f(x) = ax^2 + bx + c$$

has positive integer coefficients $a < b < c$ and satisfies:

- $f(0) = 49$ and $f(-7) = 210$
- $f(b) - f(a)$ is prime.

Find $f(1)$.

Solution. The condition $f(0) = 49$ forces $c = 49$. Next, we compute

$$f(b) - f(a) = (ab^2 + b^2 + c) - (a^3 + ab + c) = a(b^2 - a^2) + b(b-a) = (b-a)(a(b+a) + b).$$

Since $f(b) - f(a)$ is prime, one of $(b-a)$ or $(a(b+a)+b)$ must equal 1, and since both are positive and $b-a$ is the smaller of the two, we must have $b - a = 1$. Thus our polynomial takes the form

$$f(x) = ax^2 + (a+1)x + 49$$

and plugging in $x = -7$ gives $f(-7) = 49a - 7(a+1) + 49 = 210$, or $42a + 42 = 210$, giving $a = 4$. Thus $f(x) = 4x^2 + 5x + 49$, so

$$f(1) = 4 + 5 + 49 = \boxed{58.}$$

We can check that $a(b+a) + b = 4(9) + 5 = 41$ is prime. □

Problem 3. A 2-player game begins with a stack of n coins. Players alternate taking coins from the stack, with the restriction that they can only take a prime number of coins each turn. The first player who can't make a legal move (that is, there are 0 or 1 coins remaining) loses. For example, if $n = 6$, then the first player wins immediately by taking 5 coins, after which the second player has no legal moves. The smallest integer $n \geq 2$ for which the *second* player can guarantee a victory with best play is $n = 9$, and the second smallest is $n = 10$. What is the third smallest?

Solution. The answer is $\boxed{n = 25.}$ Clearly Player 1 can automatically win if n is either prime or one more than a prime, by taking n or $n - 1$ coins respectively. This gives that $n = 11$, 12, 13, 14, 17, 18, 19, 20, 23, and 24 are winning starting values for Player 1. For $n = 15$ and $n = 16$, Player 1 can take 5 or 7 coins, respectively, leaving 10 or 9 coins with Player 2, a losing position. The same argument applies to $n = 21$ and $n = 22$, when Player 1 can take 11 or 13 coins in either case, leaving Player 2 again with a losing pile of 10 or 9 coins. For $n = 25$, Player 1 cannot take 24 or 25 coins (to leave Player 2 with 0 or 1), or 15 or 16 coins (to leave Player 2 with 9 or 10), and so regardless of what Player 1 takes, Player 2 has a number of coins which we've already established is a win. □

Problem 4. Twelve cards, numbered from 1-12, are distributed, three each, to four logicians (who cannot see each others' cards). They make the statements below in order. Each statement is true and logically deducible by that logician from their cards and the previous statements.

Johnny:	At least two of my cards are prime.
Ben:	The product of my cards is even.
Reed:	The product of two my cards is the third. Also, Sue and I don't have the same number of cards with even numbers on them.
Sue:	I have a card that is numbered 2 more than Reed's smallest card.

What is the sum of Reed's cards?

Proof. Johnny's statement is irrelevant. Ben's statement means that he has at least one even card. Because two of Reed's cards multiply to the third, the possible triples of cards he has are

$$(2, 3, 6), \quad (2, 4, 8), \quad (2, 5, 10), \quad (2, 6, 12), \quad \text{or} \quad (3, 4, 12).$$

In any of these cases, observe that Reed must have 2 or 3 even cards. There are 6 even cards total. If Reed had only two even cards, then he could not deduce that he and Sue had a different number of even cards on them; it could be that Johnny and Ben each had one, and Reed and Sue each had two. So it must be that Reed has three even cards, and he deduced that Sue had at most 2 of the remaining even cards.

Eliminating triples from the above list with only two even numbers, Reed must have either $(2, 4, 8)$ or $(2, 6, 12)$. Sue, being a logician, has deduced this, and knows that Reed's smallest card is a 2. Her statement reveals that she holds the 4, meaning that Reed must have $(2, 6, 12)$. The sum of his cards is thus $\boxed{20.}$ □

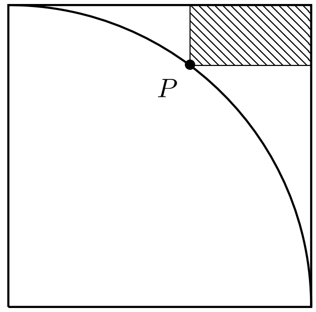
Problem 5. The figure below shows a quarter-circle inscribed in a square. A point P is drawn on the circle, and a shaded rectangle with edges parallel to those of the square.
If the shaded rectangle has area 25 and perimeter 52, find the radius of the circle.

Solution. Let the shaded rectangle have width w and height h. Since $wh = 25$ and $w + h = 26$, w and h are the roots of the quadratic polynomial $t^2 - 26t + 25 = 0$, so one is 25 and the other is 1.

Let r be the radius of the circle. Then r is the distance from P to the center of the circle, so by the Pythagorean Theorem, we have

$$(r - 25)^2 + (r - 1)^2 = r^2.$$

This simplifies to $r^2 - 52r + 626 = 0$, and the quadratic formula gives $r = 26 \pm 5\sqrt{2}$. Since $r > 25$, the only possible solution is $\boxed{r = 26 + 5\sqrt{2}}$. □



Problem 6. Suppose $f(x)$ is a function such that

$$f(x) = 3f\left(\frac{1}{1-x}\right) - 1$$

for all real numbers x except 0 and 1. Find $f(2016)$.

Solution. In fact, by applying that identity three times, we have for any real number x that

$$\begin{aligned} f(x) &= 3f\left(\frac{1}{1-x}\right) - 1 \\ &= 3\left(3f\left(\frac{1}{1-\frac{1}{1-x}}\right) - 1\right) - 1 \\ &= 9f\left(\frac{x-1}{x}\right) - 4 \\ &= 9\left(3f\left(\frac{1}{1-\frac{x-1}{x}}\right) - 1\right) - 4 \\ &= 27f(x) - 13. \end{aligned}$$

We deduce that $26f(x) = 13$ for all real numbers, so $\boxed{f(2016) = \frac{1}{2}.}$ □

Problem 7. Evaluate

$$\sqrt{142857 \times 90909 \times 77}$$

Solution. The key observation is that $\frac{1}{7} = .\overline{142857}$ and $\frac{1}{11} = \overline{.09} = \overline{.090909}$, and the numbers in the problem are very close to these same digit sequences. More explicitly, we have

$$142857 = 0.\overline{142857} \times 10^6 - 0.\overline{142857} = \frac{10^6}{7} - \frac{1}{7} = \frac{999999}{7}.$$

Similarly,

$$90909 = 0.\overline{090909} \times 10^6 - 0.\overline{090909} = \frac{10^6}{11} - \frac{1}{11} = \frac{999999}{11}.$$

We conclude

$$\sqrt{142857 \times 90909 \times 77} = \sqrt{\frac{999999}{7} \times \frac{999999}{11} \times 7 \times 11} = \sqrt{999999^2} = \boxed{999999.}$$

□

Problem 8. Below, each of the letters appearing stands for a different numeral from 0 to 9, in such a way that the resulting addition is true.

$$\begin{array}{ccccc} & M & A & T & H \\ + & & D & A & Y \\ \hline F & I & E & L & D \end{array}$$

For instance, MATH is a 4-digit integer with thousands digit M, hundreds digit A, tens digit T, and ones digit H. Note that no positive integer has 0 as its left-most digit.

What 4-digit integer is TIME?

Solution. Since $F \neq 0$, we must carry 1 from the thousands column and have $M = 9$, $F = 1$, and $I = 0$. This leaves the elements

$$(*) \qquad \{2, 3, 4, 5, 6, 7, 8\}$$

for the remaining 7 letters. As it turns out, the values of H, A, D, and Y cannot be deduced from the given information, but it is possible to deduce that $T = 5$ and $E = 2$, making the answer

$$\boxed{\text{TIME} = 5092.}$$

We carry out the case-work below. Namely, there must be a carry from the hundreds column, and we break into cases depending on carries in the ones and tens column:

- Case 1: No carries occur from either the ones or tens column. The sets of three elements from $(*)$ in which two elements sum to the third are

$$(**) \qquad \{2, 3, 5\}, \{2, 4, 6\}, \{2, 5, 7\}, \{2, 6, 8\}, \{3, 4, 7\}, \text{ and } \{3, 5, 8\}.$$

  If there are no carries in either the ones column or the tens column, then both $\{T, A, L\}$ and $\{H, Y, D\}$ must be sets in the list above, and can have no members in common, leaving E to be the remaining number. There are only two possibilities for this:
  - $\{T, A, L\}$ and $\{H, Y, D\}$ are $\{2, 4, 6\}$ and $\{3, 5, 8\}$ in some order, and $E = 7$.
  - $\{T, A, L\}$ and $\{H, Y, D\}$ are $\{2, 6, 8\}$ and $\{3, 4, 7\}$ in some order, and $E = 5$.

  Now since there are no carries in the hundreds column it must be that $A + D = 10 + E$. In all of the possibilities above, there is no way to choose one element from each of the two triples to be A and D in such a way that $A + D = 10 + E$.

- Case 2: There are carries in the tens column but not in the ones column. Then the triple $\{H, Y, D\}$ must come from $(**)$, and the triple $\{T, A, L\}$ must have the sum of two its elements be ten more than the third, with the following options:

$$(***) \qquad \{4,8,2\},\ \{5,7,2\},\ \{5,8,3\},\ \{6,7,3\},\ \{6,8,4\},\ \text{ and } \{7,8,5\}$$

  The possibilities for nonintersecting sets from $(**)$ and $(***)$ are the following options:

$\{T, A, L\}$	$\{H, Y, D\}$	E
$\{5,8,3\}$	$\{2,4,6\}$	7
$\{6,8,4\}$	$\{2,3,5\}$	7
$\{6,8,4\}$	$\{2,5,7\}$	3
$\{7,8,5\}$	$\{2,4,6\}$	3

  Now since there is a carry in the hundreds column, we have $A + D + 1 = 10 + E$. Similarly to Case 1, we find none of the options in the table can accommodate this extra condition.
- Case 3: There are carries in the ones column but not the tens column. Then $\{T, A, L\}$ must come from

$$\{2,3,6\}\,\{2,4,7\},\{2,5,8\},\{3,4,8\},$$

  and $\{H, Y, D\}$ from $(***)$. The only possibilities which don't intersect are given by the table.

$\{T, A, L\}$	$\{H, Y, D\}$	E
$\{2,3,6\}$	$\{7,8,5\}$	4
$\{2,4,7\}$	$\{5,8,3\}$	6
$\{2,5,8\}$	$\{6,7,3\}$	4
$\{3,4,8\}$	$\{5,7,2\}$	6

  Since there is no carry from the tens column, we must have $A + D = 10 + E$, and again check that there no options from the table which accomplish this.
- Case 4: There are carries in both the ones and tens column. Following the logic from the previous three cases, we find the analogous table:

$\{T, A, L\}$	$\{H, Y, D\}$	E
$\{4,7,2\}$	$\{5,8,3\}$	6
$\{4,8,3\}$	$\{5,7,2\}$	6
$\{5,7,3\}$	$\{4,8,2\}$	6
$\{5,7,3\}$	$\{6,8,4\}$	2
$\{5,8,4\}$	$\{6,7,3\}$	2
$\{6,7,4\}$	$\{5,8,3\}$	2

  Since there is a carry from the tens column, we must have $A+D+1 = 10+E$. Filtering away the options not satisfying the contraints, this gives four possibilities from the above table, corresponding to the following sum

$$\begin{array}{rrrrr} & 9 & 7 & 5 & 6 \\ + & & 4 & 7 & 8 \\ \hline 1 & 0 & 2 & 3 & 4 \end{array} \qquad \begin{array}{rrrrr} & 9 & 7 & 5 & 8 \\ + & & 4 & 7 & 6 \\ \hline 1 & 0 & 2 & 3 & 4 \end{array} \qquad \begin{array}{rrrrr} & 9 & 8 & 5 & 6 \\ + & & 3 & 8 & 7 \\ \hline 1 & 0 & 2 & 4 & 3 \end{array} \qquad \begin{array}{rrrrr} & 9 & 8 & 5 & 7 \\ + & & 3 & 8 & 6 \\ \hline 1 & 0 & 2 & 4 & 3 \end{array}$$

In each of the four possible sums, we have $T = 5$, $I = 0$, $M = 9$, and $E = 2$, giving the answer.

$\square$