MATH FIELD DAY 2015

Contestants' Instructions
Team Essay

1. Your team has *forty* minutes to answer this set of questions.

2. All answers must be justified with complete explanations. Your answers should be clear, grammatically correct, and mathematically precise.

3. Your team may turn in at most one answer to each question. Number the answers and submit them in order, starting each problem on a separate page. You are allowed to skip any question and proceed on to later questions. Write on one side of the paper only, and number the pages you turn in consecutively. Do not turn in the problem packets. Write your school name on each page.

4. Your team may be organized in any way you choose. For example, you may designate one team member to write every answer, or you could designate different team members to write different answers. You may also use the chalk board.

5. Only official team members may be in the room during the essay period. Team members may <u>not</u> bring books, calculators, or any other materials into the room.

6. A proctor outside the room will warn you ten minutes before your time is up.

7. Give your answers to the proctor at the end of the period to place in the inner envelope. Please erase all chalk boards in the room and put all scratch paper in the trash can, leaving the room neat.

Cages

A graph consists of points (called vertices) and line segments (called edges) having the vertices as their endpoints, where there is at least one vertex. In figures, we mark the vertices as dots and sometimes draw the edges as curved arcs instead of segments. Ignore crossings of edges at points other than vertices. Any two vertices are joined by at most one edge, and we identify an edge by its two endpoints.

For example, Figure 1 shows a graph with six vertices, three on the left and three on the right. The graph has nine edges, one joining each vertex on the left with each vertex on the right.

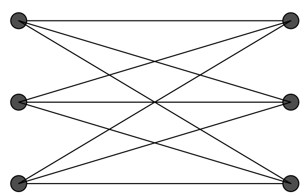

Figure 1

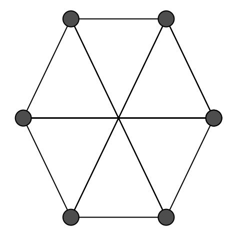

Figure 2

It does not matter how a graph is drawn. For example, Figure 2 shows the same graph as Figure 1. We can verify this by labeling the vertices in each graph so that both graphs have edges joining the same pairs of vertices. For example, when we label the vertices of Figures 1 and 2 as in Figures 3 and 4, the edges of each graph are the nine segments joining each of the points A, B, and C to each of the points D, E, and F.

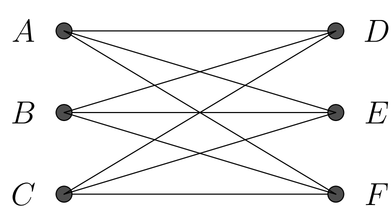

Figure 3

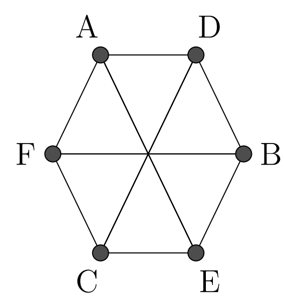

Figure 4

For any integer $n \geq 3$, a *cycle of length n* (or an *n-cycle*) is the graph formed by the vertices and sides of an n-sided polygon. For instance, Figure 5 shows the 3-cycle ABC, and Figure 6 shows the 5-cycle ABCDE. Figure 3 (or, equivalently, Figure 4) contains the 4-cycle ADBE, since it includes the four edges AD, DB, BE, and EA. It also contains the 6-cycle $AEBDCF$, since it includes the six edges AE, EB, BD, DC, CF, and FA. As always, we ignore crossings of edges at points other than their endpoints; for instance, $ADBE$ is a 4-cycle in Figure 3 even though edges DB and EA cross.

We can name an n-cycle by starting at any vertex and proceeding in either direction around the cycle. For example, the 5-cycle $ABCDE$ in Figure 6 is the same as each of the cycles $BCDEA$, $CDEAB$, $DEABC$, $EABCD$, $EDCBA$, $DCBAE$, $CBAED$, $BAEDC$, and $AEDCB$.

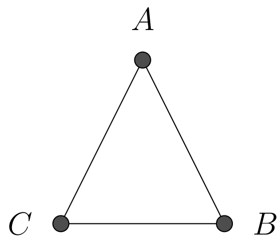

Figure 5

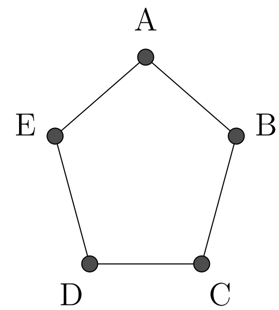

Figure 6

The *girth* of a graph is the least integer $n \geq 3$ such that the graph contains an n-cycle. For example, we claim that the graph in Figure 3 has girth 4. In fact, the graph contains the 4-cycle $ADBE$, and it includes no 3-cycles (since any three vertices in Figure 3 include at least two from the left or two from the right, and no edge joins these two vertices).

A graph is *cubic* when each vertex lies on exactly three edges. The graph in Figure 3 is cubic, since each vertex lies on the three edges joining it to the vertices on the opposite side of the figure.

Problem 1. Draw a cubic graph with four vertices. Explain why it is the only cubic graph with four vertices. Explain why every cubic graph has at least four vertices.

For any integer $n \geq 3$, an *n-cage* is a cubic graph of girth n that has as few vertices as possible. The graph in Problem 1 is the unique 3-cage, since it is a cubic graph of girth 3 and every other cubic graph has more vertices, by Problem 1. This essay determines all n-cages for $n \leq 6$.

Problem 2. Explain why every cubic graph without 3-cycles contains at least six vertices that can be labeled A-F so that the graph includes the edges in Figure 7.

Problem 3. Problem 2 and the two paragraphs before Problem 1 imply that we can get a cubic graph with no 3-cycles by adding edges (but not vertices) to Figure 7. Show that there is only one way to do so.

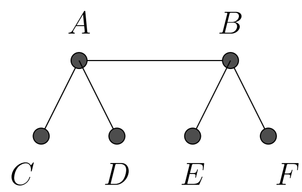

Figure 7

Problems 2 and 3 and the two paragraphs before Problem 1 show that Figure 3 is the unique 4-cage. In fact, switching the labels A and D in the answer to Problem 3 gives Figure 3.

Figure 8 shows the *Petersen graph*. It is a cubic graph with ten vertices.

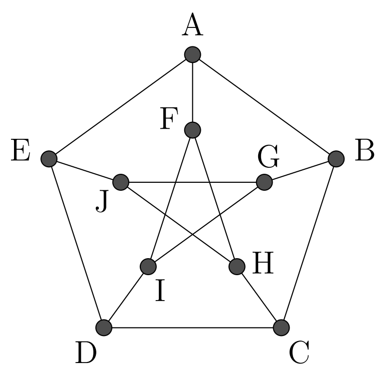

Figure 8

We can rotate Figure 8 without changing the graph so that any edge of the outer pentagon rotates to edge AB. This symmetry of Figure 8 may be useful for the next problem.

Problem 4. Show that the Petersen graph has girth 5. Give a careful argument.

Problem 5. Explain why any cubic graph with no 3-cycles and no 4-cycles includes ten vertices that can be labeled A-J so that the graph includes the edges in Figure 9.

The Petersen graph is a cubic graph with ten vertices and girth 5 (by Problem 4). Every cubic graph of girth 5 has at least ten vertices, by Problem 5. Thus, the Petersen graph is a 5-cage.

Switching the labels on G and H in Figure 9 gives the same graph, since CG and CH are the only edges on G or H. This shows one way that Figure 9 is symmetric. Use Problem 5 and symmetries of Figure 9 to do the next problem.

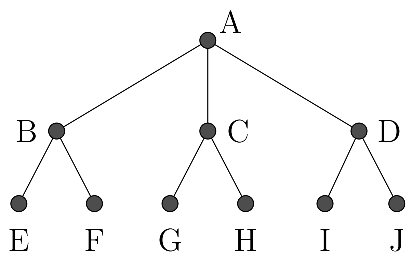

Figure 9

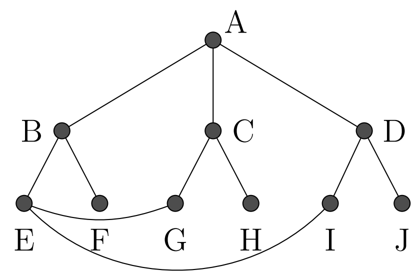

Figure 10

Problem 6. Consider a cubic graph with exactly ten vertices that has no 3-cycles and no 4-cycles. Prove that the vertices can be labeled A-J so that the graph includes the edges in Figure 10.

Problem 7. Problems 4 and 6 imply that we can get a cubic graph with no 3-cycles and no 4-cycles by adding edges (but not vertices) to Figure 10. Show that there is only one way to do so.

Problems 4-7 show that the Petersen graph is the unique 5-cage.

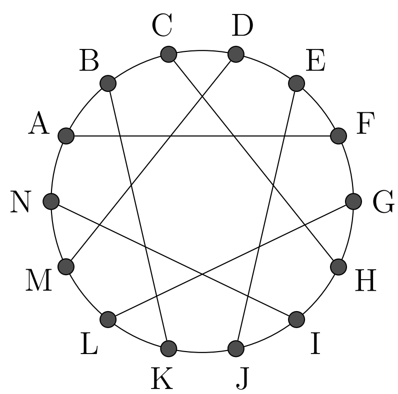

Figure 11

Figure 11 shows the *Heawood graph*. It is a cubic graph with fourteen vertices.

Rotating Figure 11 through $(360/7)^\circ$ clockwise takes arc AB to CD, BC to EF, and so on, and interchanges the seven chords of the circle in the pattern

$$AF \to CH \to EJ \to GL \to IN \to KB \to MD \to AF.$$

Thus each chord maps to AF under a rotation through a multiple of $(360/7)^\circ$, and these rotations don't change the graph. Reflecting Figure 11 across the perpendicular bisector of

A and F also leaves the graph unchanged. These symmetries of Figure 11 may be useful for the next problem.

Problem 8. Show that the Heawood graph has girth 6. Give a careful argument.

Problem 9. Consider a cubic graph that has no n-cycles for $n < 6$. Explain why the graph includes fourteen vertices that can be labeled A-N so that the graph includes the edges in Figure 12. Use Problem 2.

The Heawood graph is a cubic graph with fourteen vertices that has girth 6, by Problem 8. Every cubic graph of girth 6 has at least fourteen vertices, by Problem 9. Thus the Heawood graph is a 6-cage.

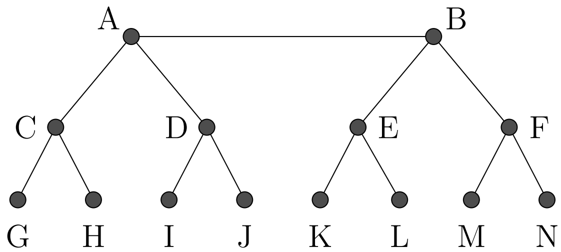

Figure 12

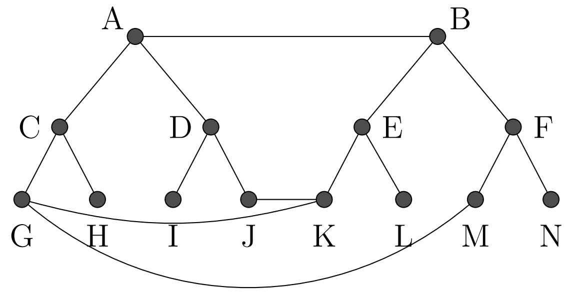

Figure 13

Problem 10. Consider a cubic graph with exactly fourteen vertices that has no n-cycles for $n < 6$. Prove that the vertices can be labeled A-N so that the graph includes the edges in Figure 13. Use Problem 9 and symmetries of Figure 12.

Problem 11. Problems 8 and 10 imply that we can get a cubic graph of girth 6 by adding edges (but not vertices) to Figure 13. Show that there is only one way to do so.

Problems 8-11 show that the Heawood graph is the unique 6-cage.

Although n-cages follow a pattern for $n \leq 6$, the pattern does not continue for $n \geq 7$. For example, Brinkmann, McKay, and Saager gave a computer-aided proof in 1995 that there are eighteen different 9-cages. All n-cages are now known for $n \leq 12$, but none are yet known for $n \geq 13$.