MATH FIELD DAY 2015

Problem 1. Draw a cubic graph with four vertices. Explain why it is the only cubic graph with four vertices. Explain why every cubic graph has at least four vertices.

In a cubic graph with four vertices, each vertex is joined by edges to the other three vertices. Then every pair of vertices is joined by an edge, and Figure 14 is the only cubic graph with four vertices.

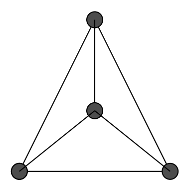

Figure 14

Any graph has at least one vertex A, by the first sentence of the Essay. If the graph is cubic, A is joined by edges to three other vertices B, C, and D. Thus, the graph has at least four vertices A-D.

Problem 2. Explain why every cubic graph without 3-cycles contains at least six vertices that can be labeled A-F so that the graph includes the edges in Figure 7.

By the answer to Problem 1, the graph includes edges AB, AC, and AD for four vertices A-D. The graph cannot include edges BC or BD, since they would give the 3-cycles ABC and ABD. Since B lies on two edges besides AB, the graph has edges BE and BF for two vertices E and F different from A-D. This gives Figure 7.

Problem 3. Problem 2 and the two paragraphs before Problem 1 imply that we can get a cubic graph with no 3-cycles by adding edges (but not vertices) to Figure 7. Show that there is only one way to do so.

To get a cubic graph by adding edges to Figure 7, we can only add edges with C, D, E, or F as endpoints, and we must add two edges on C and two on D. To get a graph of girth 4, we cannot add edge CD (since that would give the 3-cycle CDA). Thus, we must add edges CE, CF, DE, and DF, giving Figure 15. Since every vertex in Figure 15 is on three edges, we cannot add more edges, and Figure 15 is the only possibility.

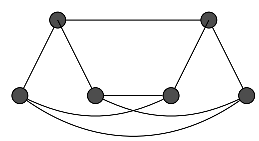

Figure 15

Problem 4. Show that the Petersen graph has girth 5. Give a careful argument.

We name the three kinds of edges in Figure 8. The "outer edges" AB, BC, CD, DE, and EA have both endpoints on the perimeter of the figure. The "diagonal edges" AF, BG, CH, DI, and EJ have one endpoint on the perimeter. The "inner edges" FH, HJ, JG, GI, and IF have no endpoints on the perimeter.

We claim that every cycle T in the Petersen graph has length at least 5. Because no two diagonal edges have a common endpoint, the only cycle without an inner edge is the 5-cycle $ABCDE$, and the only cycle without an outer edge is the 5-cycle $FHJGI$. Thus, we can assume T includes both an inner and an outer edge.

It follows that T has a string of one more consecutive outer edges preceded and followed by diagonal edges. If T has two or more outer edges, its length is at least 5 (since it has at least two diagonal edges and at least one inner edge). Thus, we can assume T has one outer edge preceded and followed by diagonal edges.

Because we can rotate Figure 8 without changing the graph, we can assume that T has the outer edge AB preceded and followed by the diagonal edges AF and BG. Since Figure 8 does not include edge FG, T has at least five edges.

We've shown that the Petersen graph has 5-cycles and that every cycle has length at least 5. Thus, the Petersen graph has girth 5.

Problem 5. Explain why any cubic graph with no 3-cycles and no 4-cycles includes ten vertices that can be labeled A-J so that the graph includes the edges in Figure 9.

By the second paragraph of the answer to Problem 1, the graph has four vertices A-D and edges AB, BC, AC, and AD, as in Figure 16.

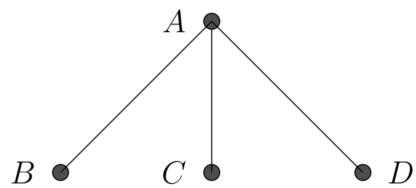

Figure 16

Let X and Y be two of the vertices B, C, and D. Because the graph has edges AX and AY, adding edge XY would give the 3-cycle AXY, and adding edges XZ and YZ for any vertex Z besides A, X, and Y would give the 4-cycle $AXZY$.

Because the graph is cubic, B, C, and D are each joined by edges to two vertices besides A. By the previous paragraph, no two of the vertices B, C, and D are joined by an edge, and no two lie on edges with a common vertex besides A. Thus, the graph has edges BE, BF, CG, CH, DI, and DJ for vertices E-J different from each other and A-D. This gives Figure 9.

Problem 6. Consider a cubic graph with exactly ten vertices that has no 3-cycles and no 4-cycles. Prove that the vertices can be labeled A-J so that the graph includes the edges in Figure 10.

By Problem 5, the ten vertices of the graph can be labeled A-J so that the graph includes the edges in Figure 9. Because A-D each lie on three edges and E-J each lie on one edge in Figure 9, we must add edges with endpoints E-J to give two more edges on each vertex E-J. We cannot add edges that create a 3-cycle or 4-cycle.

Adding edge EF would create the 3-cycle EFB. Thus, we must add two of the edges EG, EH, EI, and EJ. Adding both EG and EH would create the 4-cycle $EGCH$. Likewise, adding both EI and EJ would create the 4-cycle $EIDJ$. Thus, we must add one of the edges EG and EH and one of the edges EI and EJ.

Because the graph in Figure 9 remains unchanged when we switch G with H and when we switch I with J, we can assume that edges EG and EI are added to Figure 9. This gives Figure 10.

Problem 7. Problems 4 and 6 imply that we can get a cubic graph with no 3-cycles and no 4-cycles by adding edges (but not vertices) to Figure 10. Show that there is only one way to do so.

For F to be on three edges and no vertex to be on more that three, we must add two of the edges FG, FH, FI, and FJ to Figure 10. Adding FG would give the 4-cycle $FGEB$, and adding FI would give the 4-cycle $FIEB$. Thus, we must add edges FH and FJ to Figure 10, giving Figure 17.

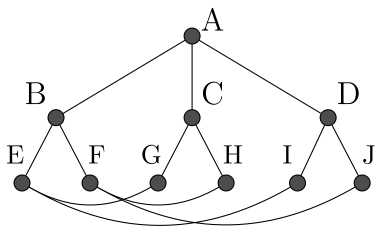

Figure 17

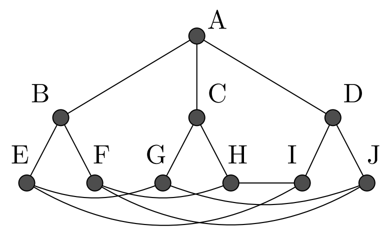

Figure 18

To get a cubic graph from Figure 17, we must add edges joining the vertices G-J in two pairs. Adding edge GH would give the 3-cycle GHC, and adding GI would give the 3-cycle GIE. Thus, we must add edges GJ and HI to Figure 17, giving Figure 18. Since every vertex in Figure 18 is on three edges, we cannot add more edges, and Figure 18 is the only possibility.

Problem 8. Show that the Heawood graph has girth 6. Give a careful argument.

We claim that any cycle T in the Heawood graph has length at least 6. If T includes no chords, it is the 14-cycle of arcs around the circumference of Figure 11.

Suppose that T includes exactly one chord. Rotating Figure 11 lets us assume that the chord is AF. Then T is either the 6-cycle $ABCDEF$ or the 10-cycle $ANMLKJIHGF$.

Suppose that T includes exactly two chords. Rotating Figure 11 lets us assume that one of the chords is AF. Let the other chord have vertices X and Y, where T includes a sequence of arcs joining A to X and a sequence of arcs joining F to Y. Each sequence has at least one arc, since no two chords share a vertex. If both sequences have at least two arcs, the length of T is at least 6 (since T has two chords and at least four arcs). Thus, we can assume that one of the sequences is a single arc. Because we can switch A with F by reflecting Figure 11 across the perpendicular bisector of A and F, we can assume that AX is an arc. Then X is either B or N, and T is either the 8-cycle $ABKJIHGF$ or the 6-cycle $ANIHGF$.

Suppose that T includes at least three chords. Then T also includes at least three arcs, since no two chords have a common endpoint. Thus, the length of T is at least six.

In short, T has 6-cycles and no cycles of length less than 6. Thus, T has girth 6.

Problem 9. Consider a cubic graph that has no n-cycles for $n < 6$. Explain why the graph includes fourteen vertices that can be labeled A-N so that the graph includes the edges in Figure 12. Use Problem 2.

By Problem 2, the graph contains the edges in Figure 7 for six vertices A-F. Any two of the vertices C-F are joined by a sequence of two or three edges in Figure 7. In fact, edges CA and AD join C to D; CA, AB, and BE join C to E; CA, AB, and BF join C to F; DA, DB, and BE join D to E; DA, AB, and BF join D to F; and EB and BF join E to F.

Let X and Y be any two of the vertices C-F. Adding XY to Figure 7 would give a 3-cycle or 4-cycle by combining XY with the sequence of two or three edges in Figure 7 that joins X to Y. Adding edges XZ and YZ to Figure 7 for a vertex Z other than A-F would create a 4-cycle or 5-cycle by combining XZ and YZ with the sequence of two or three edges in Figure 7 that joins X to Y.

In Figure 7, A and B each lie on three edges, and C-F each lie on one edge. To get a cubic graph, we must join each vertex C-F to two vertices other than A and B. By the previous paragraph, no two of the vertices C-F are joined by an edge, and no two lie on edges with a common vertex besides A and B. Thus, the graph has edges CG, CH, DI, DJ, EK, EL, FM, and FN for vertices G-N distinct from each other and from A-F. This gives Figure 12.

Problem 10. Consider a cubic graph with exactly fourteen vertices that has no n-cycles for $n < 6$. Prove that the vertices can be labeled A-N so that the graph includes the edges in Figure 13. Use Problem 9 and symmetries of Figure 12.

By Problem 9, the fourteen vertices of the graph can be labeled A-N so that the graph includes the edges in Figure 12. Because A-F each lie on three edges and G-N each lie on one edge in Figure 12, we must add edges with endpoints G-N so that we add two edges on each of these vertices. We cannot add edges that create a cycle of length less than 6.

Adding the edge GH would create the 3-cycle GHC, adding GI would create the 5-cycle $GIDAC$, and adding GJ would create the 5-cycle $GJDAC$. Thus, we must add two of the edges GK, GL, GM, and GN. Adding both GK and GL would create the 4-cycle $GKEL$, and adding both GM and GN would create the 4-cycle $GMFN$. Thus, we must add one of the edges GK and GL and one of the edges GM and GN.

Because the graph in Figure 12 remains unchanged when we switch K with L and when we switch M with N, we can assume that edges GK and GM are added to Figure 12. This gives Figure 19.

Adding edge KL to Figure 19 would create the 3-cycle KLE, adding KM would create the 3-cycle KMG, adding KN would create the 5-cycle $KNFBE$, and adding KH would create the 4-cycle $KHCG$. Thus, for K to lie on three edges, we must add edge KI or KJ

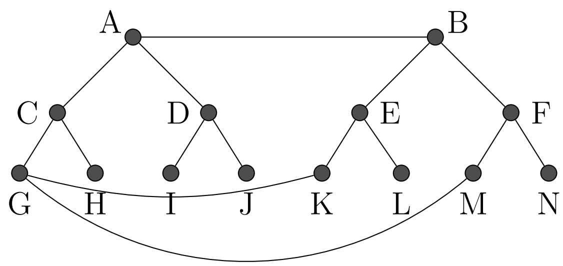

Figure 19

to Figure 19. Because Figure 19 remains unchanged when I and J are switched, we can assume that edge KJ is added to Figure 19. This gives Figure 13.

Problem 11. Problems 8 and 10 imply that we can get a cubic graph of girth 6 by adding edges (but not vertices) to Figure 13. Show that there is only one way to do so.

For H to lie on three edges and no vertex to lie on more than three edges, we must add two of the edges HI, HJ, HL, HM, and HN to Figure 13. Adding HI would give the 5-cycle $HIDAC$, adding HJ would give th 5-cycle $HJDAC$, and adding HM would give the 4-cycle $HMGC$. Thus, we must add edges HL and HN to Figure 13. This gives Figure 20.

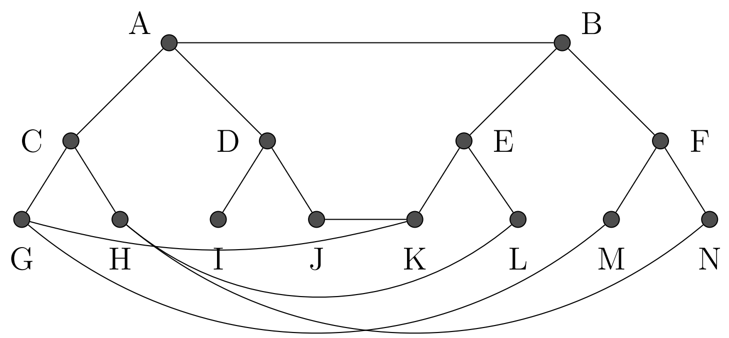

Figure 20

For L to lie on three edges and no vertex to lie on more than three edges, we must add one of the edges LI, LJ, LM, or LN to Figure 20. Adding LJ would give the 4-cycle $LJKE$, adding LM would give the 5-cycle $LMFBE$, and adding LN would give the 5-cycle $LNFBE$. Thus, we must add edge LI to Figure 20. This gives Figure 21.

To get a cubic graph from Figure 21, we must add edges joining I, J, M, and N in two pairs. Adding IJ would create the 3-cycle IJD, and adding IN would create the 4-cycle $INHL$. Thus, we must add edges IM and JN to Figure 21. This gives Figure 22.

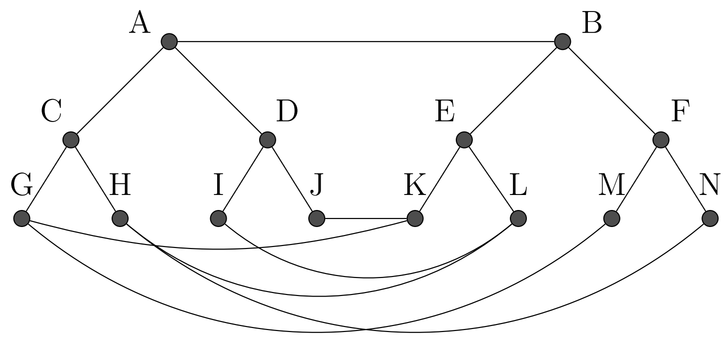

Figure 21

Since every vertex in Figure 22 lies on three edges, we cannot add more edges to Figure 22. Figure 22 is the only possibility.

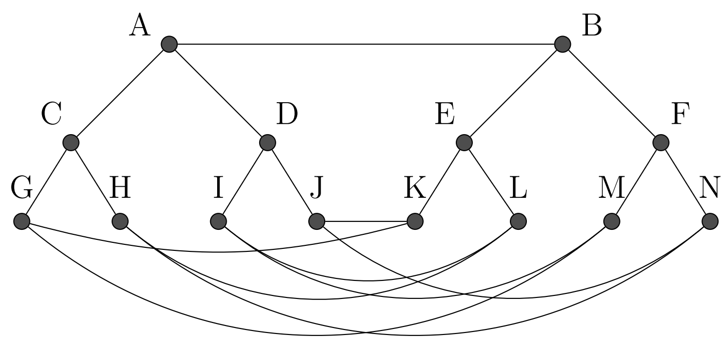

Figure 22