MATH FIELD DAY 2016

Contestants' Instructions
Team Essay

1. Your team has *forty* minutes to answer this set of questions.

2. All answers must be justified with complete explanations. Your answers should be clear, grammatically correct, and mathematically precise.

3. Your team may turn in at most one answer to each question. Number the answers and submit them in order, starting each problem on a separate page. You are allowed to skip any question and proceed on to later questions. Write on one side of the paper only, and number the pages you turn in consecutively. Do not turn in the problem packets. Write your school name on each page.

4. Your team may be organized in any way you choose. For example, you may designate one team member to write every answer, or you could designate different team members to write different answers. You may also use the chalk board.

5. Only official team members may be in the room during the essay period. Team members may not bring books, calculators, or any other materials into the room.

6. A proctor outside the room will warn you ten minutes before your time is up.

7. Give your answers to the proctor at the end of the period to place in the inner envelope. Please erase all chalk boards in the room and put all scratch paper in the trash can, leaving the room neat.

Snarks

A graph consists of points (called vertices) and line segments (called edges) whose endpoints are vertices. We mark vertices with dots. We draw graphs so that the only two vertices on an edge are its endpoints. We disregard crossings of edges at points other than vertices. Because edges are line segments, no two edges have the same two endpoints.

We require every graph to have at least one vertex. A graph is *cubic* when every vertex is an an endpoint of exactly three edges. For example, Figure 1 shows a cubic graph with eight vertices.

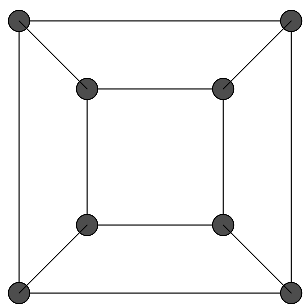

Figure 1

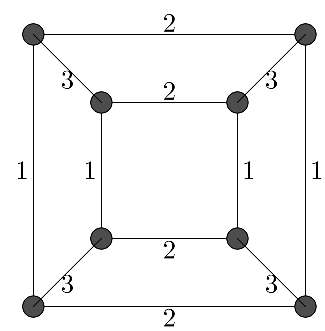

Figure 2

The edges of a cubic graph are *3-colored* when every edge is assigned one of three colors so that each vertex lies on one edge of each color. We represent the three colors by the numbers 1, 2, and 3. For example, Figure 2 shows a 3-coloring of Figure 1.

Problem 1. Give 3-colorings of the edges of the cubic graphs in Figures 3 and 4. No explanation is needed.

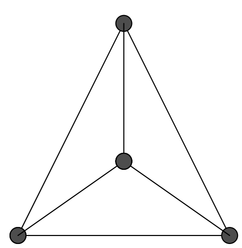

Figure 3

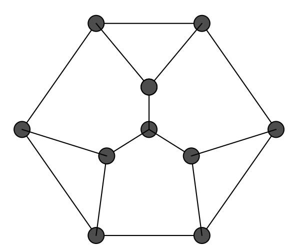

Figure 4

Figure 5 shows the *Petersen graph*. It is a cubic graph with ten vertices.

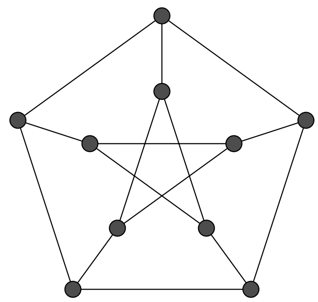

Figure 5

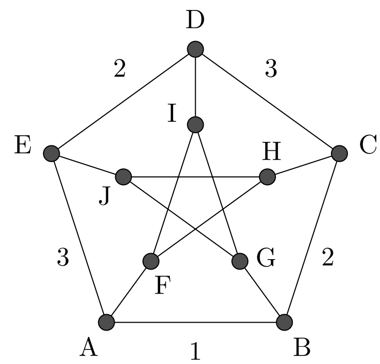

Figure 6

Figure 6 shows a choice of colors for the edges of the outer pentagon of the Petersen graph. The vertices are labeled so that edges can be identified by their endpoints; for instance, edge AB has color 1 and edges BC and DE have color 2.

Problem 2. Show that edges of the Petersen graph cannot be 3-colored so that the edges of the outer pentagon are colored as Figure 6.

We can rotate Figure 5 without changing the graph so that any edge of the outer pentagon rotates to the bottom of the figure. Use this symmetry of the Petersen graph together with Problem 2 to do the next problem.

Problem 3. Show that the edges of the Petersen graph cannot be 3-colored.

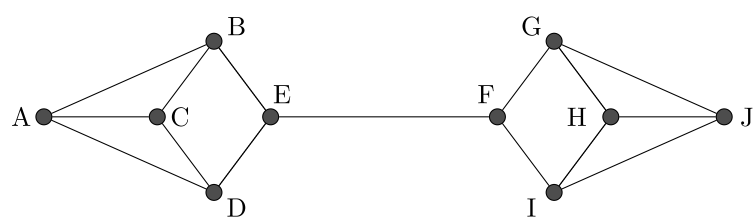

Figure 7

Figure 7 shows another cubic graph with ten vertices.

Problem 4. Prove that the edges of the graph in Figure 7 cannot be 3-colored.

We consider a graph to be unchanged when its vertices are moved, as long as the same pairs of vertices remain endpoints of edges. The rest of this essay shows that the Petersen graph and the graph in Figure 7 have fewer vertices than any other cubic graph whose edges cannot be 3-colored.

A *component* of a graph consists of all vertices joined to each other by sequences of edges and all edges with these vertices as endpoints. For example, the graph in Figure 8 has two components, one with vertices A-E and one with vertices F-J. The graph in Figure 7 has one component.

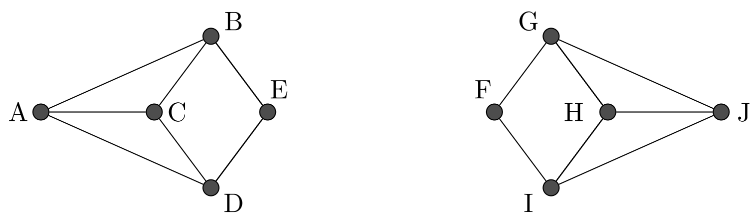

Figure 8

A *bridge* in a graph is an edge XY whose removal leaves the vertices X and Y in different components. For example, the edge EF in Figure 7 is a bridge, as Figure 8 shows. The Petersen graph has no bridges.

Graph G

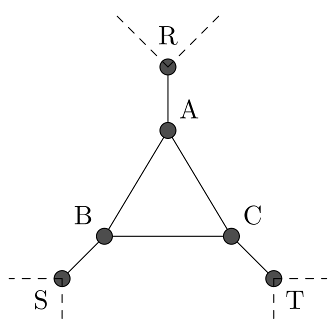

Figure 9

Graph H

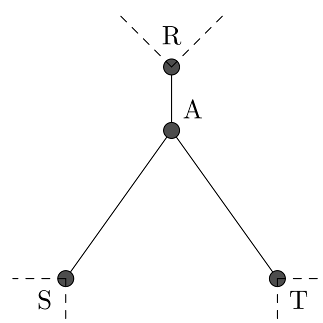

Figure 10

A graph has a *3-cycle* ABC if it includes the three vertices A, B, and C, and the three edges AB, BC, and CA.

Problems 5-7 use the following notation. Let G be a cubic graph that has a 3-cycle ABC (Figure 9). Because G is cubic, it has edges AR, BS, and CT for vertices R, S, and T not equal to A, B, or C.

Problem 5. Assume that R, S, and T are three different vertices (Figure 9). We get a cubic graph H from G by removing the vertices B and C and the five edges on them and adding edges AS and AT (Figure 10). If the edges of H can be 3-colored, show that the edges of G can also be 3-colored.

Problems 6 and 7 use the notation of the paragraph before Problem 5 with the added assumption that T equals R but not S. Then G has 3-cycles ABC and ACT but no edge BT (Figure 11). Since G is cubic, it has an edge TU for a vertex U not equal to A, C, or T. We also have $U \neq B$ (since G does not have edge BT because $S \neq T$).

Graph G

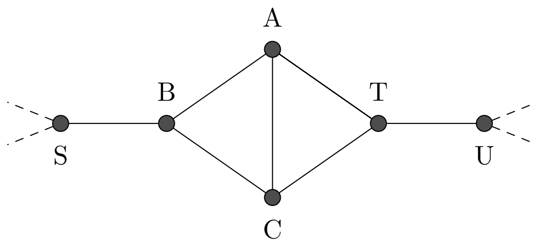

Figure 11

Graph H

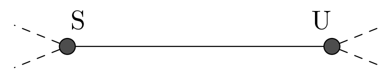

Figure 12

Problem 6. Assume that $U \neq S$ and G does not include edge SU (Figure 11). We get a cubic graph H from G by removing the vertices A, B, C, and T and the seven edges on them, and adding edge SU (Figure 12). If the edges of H can be 3-colored, show that the edges of G can also be 3-colored.

Graph G

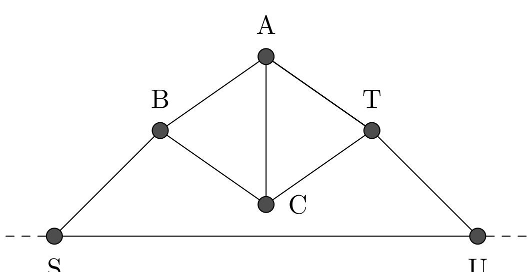

Figure 13

Graph H

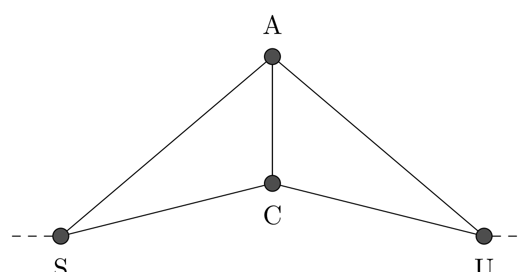

Figure 14

Problem 7. Assume that $U \neq S$ and that G includes edge SU (Figure 13). We get a cubic graph H from G by removing the vertices B and T and the six edges on them, dropping edge SU, and adding edges SA, SC, UA, and UC (Figure 14). If H can be 3-colored, show that G can be 3-colored.

Do the next problem by using Problems 5-7 and considering two possibilities we have not yet treated: that $R = S = T$ in the notation of the paragraph before Problem 5, and that $U = S$ in the notation of the paragraph before Problem 6.

Problem 8. Let G be a cubic graph whose edges cannot be 3-colored. If G has a 3-cycle ABC, prove that either G has a bridge or there is a cubic graph H that has fewer vertices than G and whose edges cannot be 3-colored.

A graph has a *4-cycle* $ABCD$ if it includes the four vertices A, B, C, and D and the four edges AB, BC, CD, and DA.

Problems 9 and 10 use the following notation. Let G be a cubic graph that has a 4-cycle $ABCD$ and has no 3-cycles (Figure 15). Since G is cubic, it has edges AR and BS for a vertex R not equal to A, B, or D and a vertex S not equal to A, B, or C. Since G has no 3-cycles, we also have $R \neq C$, $S \neq D$, and $R \neq S$, so A-D, R, and S are six different points.

Problem 9. Assume that G contains neither edge CR nor DS (Figure 15). We get a cubic graph H from G by removing the vertices A and B and the five edges on them and adding edges CR and DS (Figure 16). If the edges of H can be 3-colored, show that the edges of G can also be 3-colored. Take into account the possibility of coloring edges in Figure 16 in ways that differ by more than interchanging the three colors.

Graph G

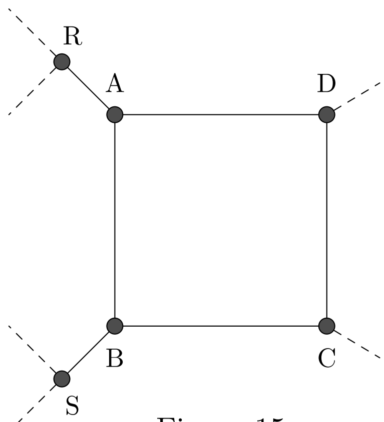

Figure 15

Graph H

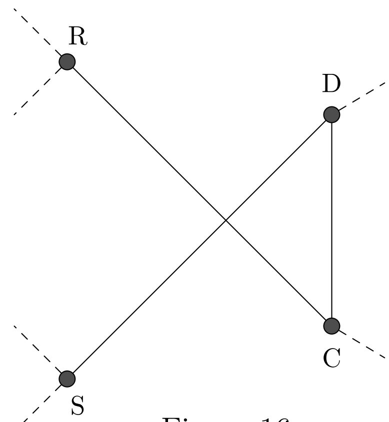

Figure 16

Problem 10. Assume that G contains edge CR (Figure 17). Since G has no 3-cycles, it includes none of the edges BR, DR, and BD. We get a cubic graph H from G by removing the vertices A and C and the six edges on them and adding edges BR, DR, and BD (Figure 18). If the edges of H can be 3-colored, show that the edges of G can also be 3-colored.

Graph G

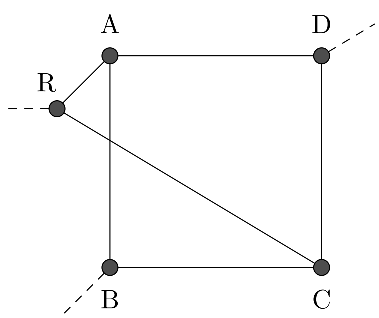

Figure 17

Graph H

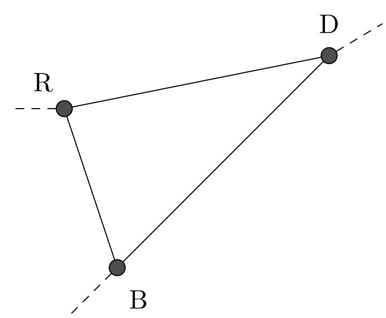

Figure 18

The 2015 Team Essay included the following result: the Petersen graph has no 3-cycles and no 4-cycles, and it has fewer vertices than any other cubic graph with no 3-cycles and no 4-cycles.

Problem 11. Let G be a cubic graph whose edges cannot be 3-colored and that has no bridges. Prove that G either is the Petersen graph or has more vertices than another cubic graph whose edges cannot be 3-colored. Use Problems 8-10, the result from the 2015 Team Essay, and Problem 3.

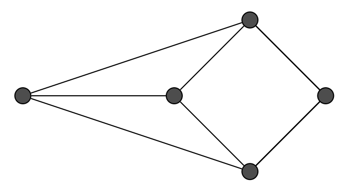

Figure 19

Problem 12. Let a cubic graph G have a bridge EF. Consider the component K containing E when EF is removed from G. Prove that K either is the graph in Figure 19 or has at least 6 vertices.

Problem 13. Prove that the Petersen graph and the graph in Figure 7 have fewer vertices than any other cubic graph whose edges cannot be 3-colored. Use Problems 3, 4, 11, and 12.

A *snark* is a cubic graph whose edges cannot be 3-colored and that has no bridges. We've proved that the Petersen graph is a snark with fewer vertices than any other. (Snarks are usually defined in a more restrictive way in order to focus on the most interesting examples.)

W.T. Tutte conjectured in 1967 that every snark results from subdividing the edges of the Petersen graph with new vertices and adding more vertices and edges. Efforts to find a computer-aided proof of Tutte's conjecture have not yet been completed.

The mathematician Charles Dodgson wrote Alice's Adventures in Wonderland under the pen name Lewis Carroll. He invented the name "snark" for an elusive wild creature in his nonsense poem "The Hunting of the Snark." Martin Gardner, an expert writer on both mathematics and Lewis Carroll, transferred the name "snark" to an elusive type of graph.