MATH FIELD DAY 2016

Problem 1. Give 3-colorings of the edges of the cubic graphs in Figures 3 and 4. No explanation is needed.

Solution. Solution below:

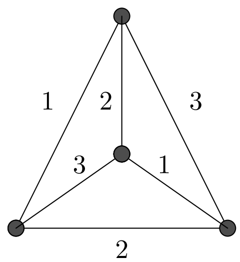

Figure 20

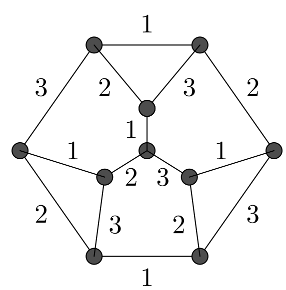

Figure 21

□

Problem 2. Show that edges of the Petersen graph cannot be 3-colored so that the edges of the outer pentagon are colored as Figure 6.

Solution. Since each vertex of the outer pentagon is already on edges of two colors, the third edge on the vertex must be the third color. This gives AF color 2, BG color 3, and CH, DI, and EJ color 1 (Figure 22). Since edges with a common vertex are different colors, FH has color 3, and FI cannot be colored. □

Problem 3. Show that the edges of the Petersen graph cannot be 3-colored.

Solution. Suppose the edges of the Petersen graph could be 3-colored. The outer pentagon cannot have three edges of the same color, since two of the edges would have a common vertex. Thus, the pentagon has one edge of one color and two of each of the other two colors. Label the colors so that the pentagon has one edge of color 1 and two of each of the

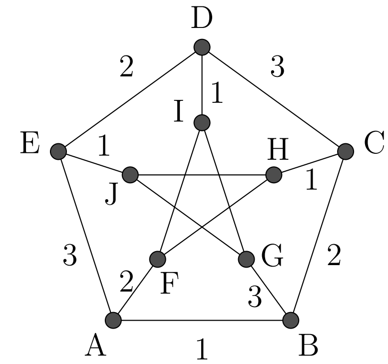

Figure 22

other two colors. Label the colors so that the pentagon has one edge of color 1 and the next edge counterclockwise has color 2. Rotate the pentagon to give AB color 1 and BC color 2. Since no other edge of the outer pentagon has color 1, CD has color 3, DE has color 2, and EF has color 3. This gives the coloring of edges in Figure 6, which does not extend to a 3-coloring of all edges, by Problem 2. □

Problem 4. Prove that the edges of the graph in Figure 7 cannot be 3-colored.

Solution. Suppose that the edges of Figure 7 are 3-colored. Because the three edges on A are different colors, label the color of AB as 1, AC as 2, and AD as 3 (Figure 23). Since edges with a common vertex have different colors, BC has color 3, BE has color 2, CD has color 1, and DE cannot be colored.

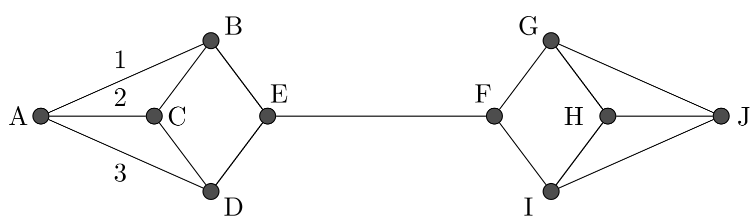

Figure 23

□

Problem 5. Assume that R, S, and T are three different vertices (Figure 9). We get a cubic graph H from G by removing the vertices B and C and the five edges on them and adding edges AS and AT (Figure 10). If the edges of H can be 3-colored, show that the edges of G can also be 3-colored.

Solution. If the edges of H are 3-colored, the three edges on A are different colors. Label the colors as in Figure 24. Color edges of G as in Figure 25, giving all other edges of G the same colors as in H. Figure 25 shows that A, B, and C are each on edges of three different colors. All other vertices of G are on edges of the same three colors as in H. Thus, the edges of G are 3-colored.

Graph H

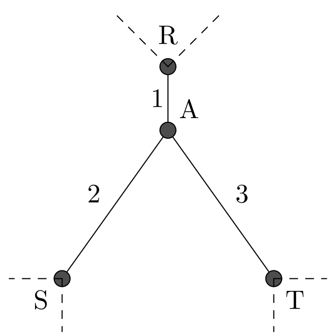

Figure 24

Graph G

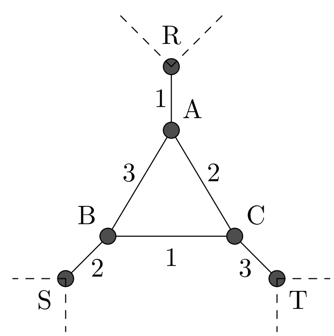

Figure 25

□

Problem 6. Assume that $U \neq S$ and G does not include edge SU (Figure 11). We get a cubic graph H from G by removing the vertices A, B, C, and T and the seven edges on them, and adding edge SU (Figure 12). If the edges of H can be 3-colored, show that the edges of G can also be 3-colored.

Solution. If the edges of H are 3-colored, label the color of the edge SU as 1. Color edges of G as in Figure 26, giving the other edges of G the same colors as in H. In Figure 26, A, B, C, and T are each on edges of three different colors. All other vertices of G are on edges of the same three colors as in H. Thus, the edges of G are 3-colored.

□

Graph G

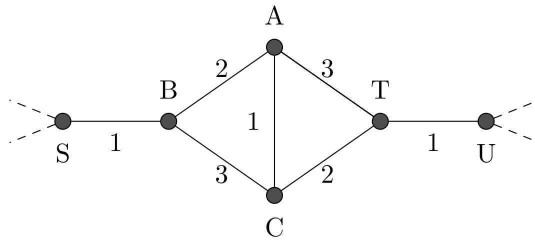

Figure 26

Problem 7. Assume that $U \neq S$ and that G includes edge SU (Figure 13). We get a cubic graph H from G by removing the vertices B and T and the six edges on them, dropping edge SU, and adding edges SA, SC, UA, and UC (Figure 14). If H can be 3-colored, show that G can be 3-colored.

Solution. If the edges of H are 3-colored, the three edges on A are different colors. Label the color of AC as 1, AS as 2, and AU as 3 (Figure 27). Since edges with a common vertex are different colors, CS has color 3, and CU has color 2.

Color edges of G as in Figure 28, giving the other edges of G the same colors as in H. In Figure 28, A, B, C, and T are each on edges of three different colors. All other vertices of G are on edges of the same three colors as in H. Thus, the edges of G are 3-colored.

Graph H

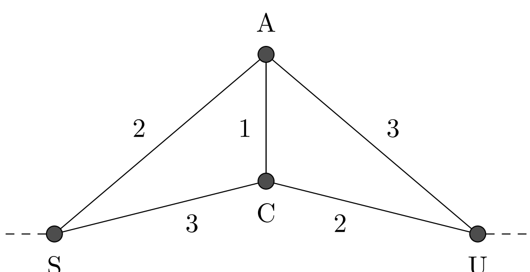

Figure 27

Graph G

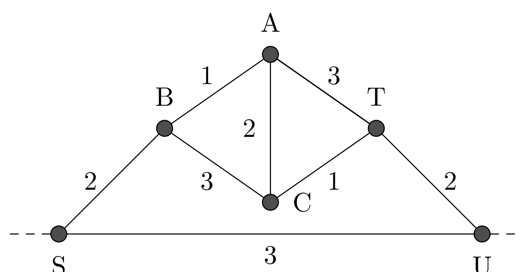

Figure 28

□

Problem 8. Let G be a cubic graph whose edges cannot be 3-colored. If G has a 3-cycle

ABC, prove that either G has a bridge or there is a cubic graph H that has fewer vertices than G and whose edges cannot be 3-colored.

Solution. G has vertices R, S, and T as before Problem 5 (Figure 9). If R, S, and T are three different points, Problem 5 gives a cubic graph H that has fewer vertices than G and whose edges cannot be 3-colored (since the edges of G cannot be 3-colored).

Suppose that exactly two of the points R, S, and T are equal. By symmetry, we can assume that T equals R but not S. G has a vertex U as before Problem 6 (Figure 11). If $S \neq U$, then, depending on whether or not G has an edge SU, either Problem 6 or Problem 7 gives a graph H that has fewer vertices than G and whose edges cannot be 3-colored (since the edges of G cannot be 3-colored). If $S = U$, then G includes the edges in Figure 29. Because G is cubic, it has exactly one more edge on U than in the figure, and that edge is a bridge of G (since adding it to Figure 29 gives three edges on each of the pictured vertices).

Finally, suppose that R, S, and T are all equal. Then G includes the edges in Figure 30. That figure is a cubic graph whose edges can be 3-colored (by Problem 1). Since G is a cubic graph whose edges cannot be 3-colored, the vertices and edges of G not in Figure 30 form a cubic graph H that has fewer vertices than G and whose edges cannot be 3-colored.

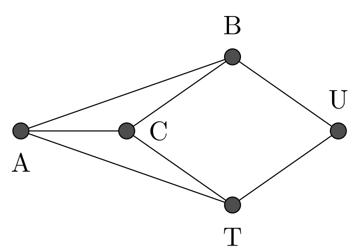

Figure 29

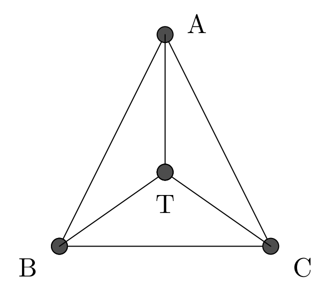

Figure 30

□

Problem 9. Assume that G contains neither edge CR nor DS (Figure 15). We get a cubic graph H from G by removing the vertices A and B and the five edges on them and adding edges CR and DS (Figure 16). If the edges of H can be 3-colored, show that the edges of G can also be 3-colored. Take into account the possibility of coloring edges in Figure 16 in ways that differ by more than interchanging the three colors.

Solution. If the edges of H are 3-colored, the edges on C are different colors. Label the color of CD as 1 and CR as 2. Because the edges on D are different colors, edge DS has either color 3 (Figure 31) or 2 (Figure 32). Color edges of G as in Figure 33 in the first case and Figure 34 in the second. In both cases, color the rest of the edges of G as in H. In both cases, A and B are each on edges of three different colors, and the other vertices of G are on edges of the same three colors as in H. Thus the edges of G are 3-colored.

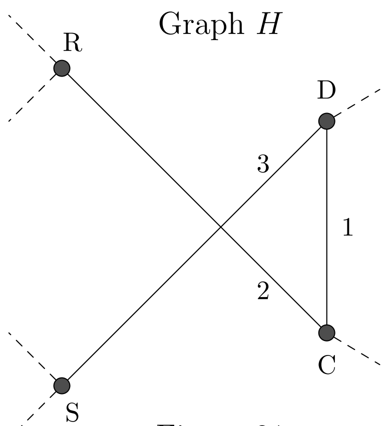

Figure 31

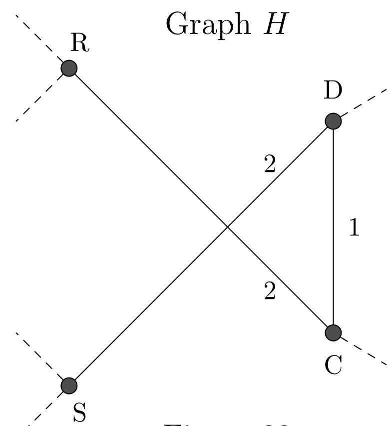

Figure 32

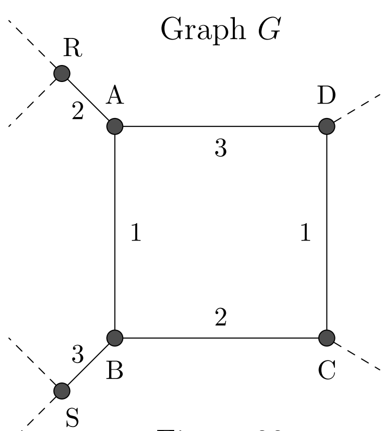

Figure 33

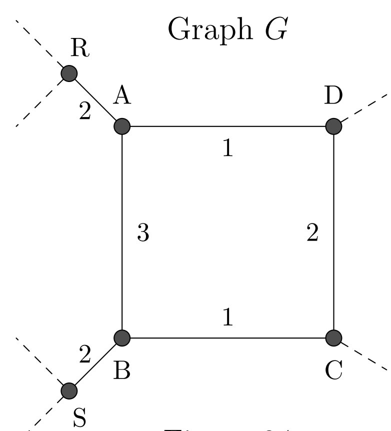

Figure 34

$\square$

Problem 10. Assume that G contains edge CR (Figure 17). Since G has no 3-cycles, it includes none of the edges BR, DR, and BD. We get a cubic graph H from G by removing the vertices A and C and the six edges on them and adding edges BR, DR, and BD (Figure 18). If the edges of H can be 3-colored, show that the edges of G can also be 3-colored.

Solution. If the edges of H are 3-colored, edges BD, BR, and DR are different colors (since any two share a vertex). Label the color of BD as 1, BR as 2, and DR as 3 (Figure 35). Color edges of G as in Figure 36, where all other edges of G have the same colors as in H. Figure 36 shows that A and C are each on edges of three different colors, and all other vertices of G are on edges of the same three colors as in H. Thus, the edges of G are 3-colored.

Graph H

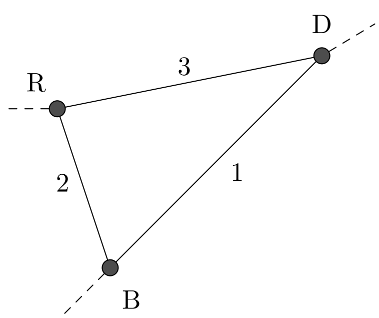

Figure 35

Graph G

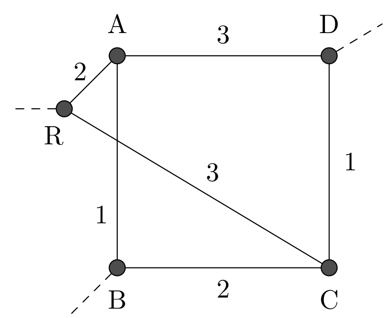

Figure 36

□

Problem 11. Let G be a cubic graph whose edges cannot be 3-colored and that has no bridges. Prove that G either is the Petersen graph or has more vertices than another cubic graph whose edges cannot be 3-colored. Use Problems 8-10, the result from the 2015 Team Essay, and Problem 3.

Solution. If G has a 3-cycle, then, since G has no bridges, Problem 8 gives a cubic graph H that has fewer vertices than G and whose edges cannot be 3-colored. Thus, we can assume that G has no 3-cycles.

Suppose that G has a 4-cycle $ABCD$. G has vertices R and S as before Problem 9 (Figure 15). If neither CR nor DS is an edge of G, Problem 9 gives a cubic graph H that has fewer vertices than G and whose edges cannot be 3-colored (since the edges of G cannot be 3-colored). If CR or DS is an edge of G, we can assume that CR is, by symmetry (Figure 17). Then Problem 10 gives a cubic graph H that has fewer vertices than G and whose edges cannot be 3-colored (since the edges of G cannot be 3-colored).

We're left with the case where G has no 3-cycles and no 4-cycles. G either is the Petersen graph or has more vertices than the Petersen graph (by the result of the 2015 Team Essay), and the Petersen graph is a cubic graph whose edges cannot be 3-colored (by Problem 3). □

Problem 12. Let a cubic graph G have a bridge EF. Consider the component K containing E when EF is removed from G. Prove that K either is the graph in Figure 19 or has at least 6 vertices.

Solution. Because G is cubic, E lies on exactly two edges BE and DE of K (Figure 37). Since B lies on three edges of K, it lies on an edge BC for a vertex C of K not equal to B, D, or E. Since K has three edges on C but no edge CE, it has an edge CA for a vertex A of K not equal to B, C, D, or E. Thus, K has at least the five vertices A-E.

Suppose that A-E are the only five vertices in K. Besides the edges in Figure 37, K has edges AB and AD (since it has three edges on A and no edge AE) and edge CD (since K has three edges on C and no edge CE). Thus, K has the edges shown in Figure 38. It has no other edges (since G is cubic), and so it's the graph in Figure 19.

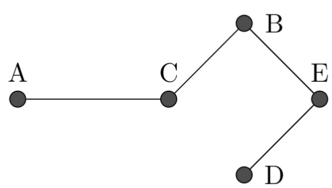

Figure 37

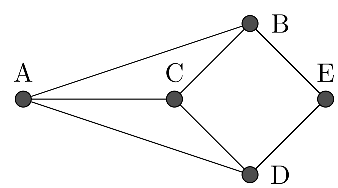

Figure 38

□

Problem 13. Prove that the Petersen graph and the graph in Figure 7 have fewer vertices than any other cubic graph whose edges cannot be 3-colored. Use Problems 3, 4, 11, and 12.

Solution. Let L be a cubic graph with a bridge EF. E and F belong to different components when edge EF is removed. By Problem 12, each component has at least five vertices, and it only has five vertices when it's the graph in Figure 19. Thus, L has at least 10 vertices, and it only has ten vertices when it's the graph in Figure 7.

The Petersen graph and the graph in Figure 7 are cubic graphs whose edges cannot be 3-colored (by Problems 3 and 4). Thus, there is a cubic graph G whose edges cannot be 3-colored and that has as few vertices as possible. If G has no bridges, it must be the Petersen graph, by Problem 11. If G has a bridge, it must be the graph in Figure 7, by the previous paragraph (since Figure 7 is a cubic graph whose edges cannot be 3-colored).

In short, the only two possibilities for G are the Petersen graph and the graph in Figure 7. Because both of these graphs have ten vertices, every other cubic graph whose edges cannot be 3-colored has more than ten vertices. □