Problem 1. If $m$ and $n$ are positive integers such that

$$mn + m + n = 2014,$$

what is the largest possible value of $m$?

Solution. We re-write the equation as $(m+1)(n+1) = 2015$, so the numbers $m+1$ and $n+1$ divide evenly into 2015. We get the largest possible value of $m$ when $n+1$ is the smallest divisor of 2015 greater than 1. Since 2, 3, and 4 do not go into 2015, the smallest is $n+1 = 5$. Then $m+1 = \frac{2015}{5} = 403$, so $m = 402$ is the largest possible $m$. □

Problem 2. Evaluate

$$
\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{2014^2}\right)
$$

Solution. Simplifying, we see that the product

$$
= \left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 + \frac{1}{3}\right)\cdots\left(1 - \frac{1}{2014}\right)\left(1 + \frac{1}{2014}\right)
$$

$$
= \frac{1 \cdot 3 \cdot 2}{2} \cdot \frac{2 \cdot 4}{3} \cdot \frac{3 \cdot 5}{4} \cdots \frac{2012 \cdot 2014}{2013} \cdot \frac{2013 \cdot 2014}{2014}.
$$

In both the numerator and denominator, each integer from 3 to 2013 appears twice. In addition, there is one factor of 2, and one of 2014, in both numerator and denominator. After cancelling these factors, we are left with

$$
\frac{2015}{2 \cdot 2014} = \frac{2015}{4028}.
$$

□

Problem 3. Suppose that $A$ and $B$ are points in the plane one unit apart. How many points $C$ are there such that $\triangle ABC$ is a right triangle with perimeter $\frac{9}{4}$? (Any of the three vertices of $\triangle ABC$ can be $90^\circ$.)

Solution. We can suppose $A = (0,0)$ and $B = (1,0)$. We split into three cases, depending on the location of the right angle.

- If the right angle is at $A$, then $C$ is on the line $x = 0$, and there are two points on that line where the perimeter is exactly $\frac{9}{4}$ (the exact $y$-value of these points is not important).
- If the right angle is at $B$, then $C$ is on the line $x = 1$ and again there are two solutions.
- If the right angle is at $C$, then $C$ must lie on the circle with diameter $AB$. To see how many points have the perimeter condition, we need to calculate the perimeter obtained by taking $C = (1/2, 1/2)$, which we quickly work out to be precisely $1 + \sqrt{2}$. Since $\frac{9}{4} < 1 + \sqrt{2}$, there are four points $C$ on the circle for which the right triangle $ABC$ has perimeter $\frac{9}{4}$.

Combining the three cases, we see there is a total of 8 possibilities. □
Problem 4. Evaluate
\[
\sqrt{(\cos(107) - \cos(17))^2 + (\sin(107) - \sin(17))^2} \\
+ \sqrt{(\cos(197) - \cos(107))^2 + (\sin(197) - \sin(107))^2} \\
+ \sqrt{(\cos(287) - \cos(197))^2 + (\sin(287) - \sin(197))^2} \\
+ \sqrt{(\cos(17) - \cos(287))^2 + (\sin(17) - \sin(287))^2}
\]

Solution. The four summands are the distances between the four equally spaced points \((\cos(17), \sin(17))\), \((\cos(107), \sin(107))\), \((\cos(197), \sin(197))\), and \((\cos(287), \sin(287))\) on the unit circle, i.e., of a square inscribed in the unit circle. Since the diagonal of such a square is 2, each side length is \(\sqrt{2}\) units, so the perimeter is \(4\sqrt{2}\). \(\square\)

Problem 5. Find \(x_5\) if \(x_1 = \log_9(3)\), \(x_2 = x_1^{\log_7(2)}\), \(x_3 = x_2^{\log_5(8)}\), \(x_4 = x_3^{\log_2(5)}\), and \(x_5 = x_4^{\log_2(49)}\).

Solution. By repeatedly substituting, we get
\[
x_5 = (x_1^{\log_7(2)})^{\log_5(8)}^{\log_2(5)}^{\log_2(49)}.
\]
Using the log rule \(\log_a(b) = \frac{\ln(b)}{\ln(a)}\) we can simplify this exponent as
\[
\frac{\ln(2)}{\ln(7)} \cdot \frac{\ln(8)}{\ln(5)} \cdot \frac{\ln(49)}{\ln(2)} = \frac{\ln(2^4)}{\ln(2)} = \frac{3 \ln(2) \cdot 2 \ln(7)}{\ln(2) \ln(7)} = 6.
\]
Thus \(x_5 = x_1^6\), and since \(x_1 = \log_9(3) = \frac{1}{2}\), we get \(x_5 = \frac{1}{64}\). \(\square\)

Problem 6. Suppose that 10 fair 8-sided dice (with sides 1, 2, 3, 4, 5, 6, 7, 8) are rolled. Find the probability that the product of the numbers rolled is prime. Write your answer as a reduced fraction in the form \(\frac{m}{8^n}\).

Solution. In order for the product to be prime, nine of the rolls would have to be a 1, and one of the rolls would have to be a 2, 3, 5, or 7. The probability that the first roll is a 2, 3, 5, or 7, and the rest are all 1’s, is simply
\[
\frac{4}{8} \left(\frac{1}{8}\right)^9.
\]
This is also the probability that the second roll is one of those values and the rest are 1’s, or the third, etc. Altogether, the total probability is then
\[
10 \cdot \frac{4}{8} \left(\frac{1}{8}\right)^9 = \frac{40}{8^{10}} = \frac{5}{8^9}.
\]
\(\square\)
Problem 7. Boise is trying to guess the polynomial $x^3 + bx^2 + cx + d$, with integer coefficients $b$, $c$, and $d$, that Salem is thinking of.

Boise: Is $c \leq 20$?
Salem: Yes.
Boise: Does the polynomial factor as $(x - r)^2(x - s)$ for positive integers $r$ and $s$?
Salem: Yes.
Boise: Could I determine $r$ and $s$ if I knew the value of $c$?
Salem: No.

Find $c$.

Solution. The equality $x^2 + bx^2 + cx + d = (x - r)^2(x - s)$ gives $c = r^2 + 2rs$. Since it is impossible to determine $r$ and $s$ from $c$, it must be that $c$ can be written in the form $r^2 + 2rs$ for two different pairs of $r$ and $s$. For $c \leq 20$, the only possibility is $c = 15$, corresponding to $r = 1$ and $s = 7$, or $r = 3$ and $s = 1$. \[\square\]

Problem 8. Suppose that $x > y > 0$ are real numbers satisfying

$$(x - y)^2 = 512$$
$$x - y = 64$$
$$x^2 - y^2 = 16$$

Find $(x + y)^2$.

Solution. We have

$$(x - y)^2 - 2xy + y^2 = \frac{(x - y)^2 \cdot (x - y)^2}{(x - y)^2} = \frac{2^9 \cdot 2^4}{(2^6)^2} = 2.$$

Squaring both sides gives

$$(x^2 - 2xy + y^2)^2 - 2xy + y^2 = 4.$$

The only positive number which when raised to itself gives 4 is 2, so we must have $2 = x^2 - 2xy + y^2 = (x - y)^2$. Since $x - y > 0$, we must have $x - y = \sqrt{2}$. Substituting into the first formula gives $\sqrt{2}^2 = 2^9 = \sqrt{2}^{18}$. This gives $x^2 = 18$, or $x = 3\sqrt{2}$. Similarly, in the third formula we get $\sqrt{2}^y = 2^4 = \sqrt{2}^8$, so $y^2 = 8$, and $y = 2\sqrt{2}$. Finally,

$$(x + y)^2 = (3\sqrt{2} + 2\sqrt{2})^2 = (5\sqrt{2})^2 = 50.$$ \[\square\]