2014 LEAP FROG PROBLEMS AND SOLUTIONS

Problem 1. If m and n are positive integers such that

$$mn + m + n = 2014,$$

what is the largest possible value of m?

Solution. We re-write the equation as $(m+1)(n+1) = 2015$, so the numbers $m+1$ and $n+1$ divide evenly into 2015. We get the largest possible value of m when $n+1$ is the smallest divisor of 2015 greater than 1. Since 2, 3, and 4 do not go into 2015, the smallest is $n+1 = 5$. Then

$$m + 1 = \frac{2015}{5} = 403,$$

so $\boxed{m = 402}$ is the largest possible m. □

Problem 2. Evaluate

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{2014^2}\right)$$

Solution. Simplifying, we see that the product

$$\begin{aligned} &= \left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 + \frac{1}{3}\right)\cdots\left(1 - \frac{1}{2014}\right)\left(1 + \frac{1}{2014}\right) \\ &= \frac{1}{2}\cdot\frac{3}{2}\times\frac{2}{3}\cdot\frac{4}{3}\times\frac{3}{4}\cdot\frac{5}{4}\times\cdots\times\frac{2012}{2013}\cdot\frac{2014}{2013}\times\frac{2013}{2014}\cdot\frac{2015}{2014}. \end{aligned}$$

In both the numerator and denominator, each integer from 3 to 2013 appears twice. In addition, there is one factor of 2, and one of 2014, in both numerator and denominator. After cancelling these factors, we are left with

$$\frac{2015}{2 \cdot 2014} = \boxed{\frac{2015}{4028}.}$$

□

Problem 3. Suppose that A and B are points in the plane one unit apart. How many points C are there such that $\triangle ABC$ is a right triangle with perimeter $\frac{9}{4}$? (Any of the three vertices of $\triangle ABC$ can be 90°.)

Solution. We can suppose $A = (0, 0)$ and $B = (1, 0)$. We split into three cases, depending on the location of the right angle.

- If the right angle is at at A, then C is on the line $x = 0$, and there are *two* points on that line where the perimeter is exactly $\frac{9}{4}$ (the exact y-value of these points is not important).
- If the right angle is at B, then C is on the line $x = 1$ and again there are *two* solutions.
- If the right angle is at C, then C must lie on the circle with diameter AB. To see how many points have the perimeter condition, we need to calculate the perimeter obtained by taking $C = (1/2, 1/2)$, which we quickly work out to be precisely $1 + \sqrt{2}$. Since $\frac{9}{4} < 1 + \sqrt{2}$, there are *four* points C on the circle for which the right triangle ABC has perimeter $\frac{9}{4}$.

Combining the three cases, we see there is a total of $\boxed{8}$ possibilities. □

Problem 4. Evaluate

$$\begin{aligned}&\sqrt{(\cos(107)-\cos(17))^2+(\sin(107)-\sin(17))^2}\\+&\sqrt{(\cos(197)-\cos(107))^2+(\sin(197)-\sin(107))^2}\\+&\sqrt{(\cos(287)-\cos(197))^2+(\sin(287)-\sin(197))^2}\\+&\sqrt{(\cos(17)-\cos(287))^2+(\sin(17)-\sin(287))^2}\end{aligned}$$

Solution. The four summands are the distances between the four equally spaced points $(\cos(17), \sin(17))$, $(\cos(107), \sin(107))$, $(\cos(197), \sin(197))$, and $(\cos(287), \sin(287))$ on the unit circle, i.e., of a square inscribed in the unit circle. Since the diagonal of such a square is 2, each side length is $\sqrt{2}$ units, so the perimeter is $\boxed{4\sqrt{2}.}$ □

Problem 5. Find x_5 if $x_1 = \log_9(3)$, $x_2 = x_1^{\log_7(2)}$, $x_3 = x_2^{\log_5(8)}$, $x_4 = x_3^{\log_2(5)}$, and $x_5 = x_4^{\log_2(49)}$.

Solution. By repeatedly substituting, we get

$$x_5 = (x_1^{\log_7(2)})^{\log_5(8)\cdot\log_2(5)\cdot\log_2(49)}.$$

Using the log rule $\log_a(b) = \frac{\ln(b)}{\ln(a)}$, we can simplify this exponent as

$$\frac{\ln(2)}{\ln(7)}\cdot\frac{\ln(8)}{\ln(5)}\cdot\frac{\ln(5)}{\ln(2)}\cdot\frac{\ln(49)}{\ln(2)} = \frac{\ln(2^3)\ln(7^2)}{\ln(2)\ln(7)} = \frac{3\ln(2)\cdot 2\ln(7)}{\ln(2)\ln(7)} = 6.$$

Thus $x_5 = x_1^6$, and since $x_1 = \log_9(3) = \frac{1}{2}$, we get $\boxed{x_5 = \frac{1}{2^6} = \frac{1}{64}.}$ □

Problem 6. Suppose that 10 fair 8-sided dice (with sides 1, 2, 3, 4, 5, 6, 7, 8) are rolled. Find the probability that the product of the numbers rolled is prime. Write your answer as a reduced fraction in the form

$$\frac{m}{8^n}.$$

Solution. In order for the product to be prime, nine of the rolls would have to be a 1, and one of the rolls would have to be a 2, 3, 5, or 7. The probability that the *first* roll is a 2, 3, 5, or 7, and the rest are all 1's, is simply

$$\frac{4}{8}\cdot\left(\frac{1}{8}\right)^9.$$

This is also the probability that the second roll is one of those values and the rest are 1's, or the third, etc. Altogether, the total probability is then

$$10\cdot\frac{4}{8}\cdot\left(\frac{1}{8}\right)^9 = \frac{40}{8^{10}} = \boxed{\frac{5}{8^9}.}$$

□

Problem 7. Boise is trying to guess the polynomial $x^3 + bx^2 + cx + d$, with integer coefficients b, c, and d, that Salem is thinking of.

Boise: Is $c \leq 20$?
Salem: Yes.
Boise: Does the polynomial factor as $(x-r)^2(x-s)$ for positive integers r and s?
Salem: Yes.
Boise: Could I determine r and s if I knew the value of c?
Salem: No.

Find c.

Solution. The equality

$$x^2 + bx^2 + cx + d = (x-r)^2(x-s)$$

gives $c = r^2 + 2rs$. Since it is impossible to determine r and s from c, it must be that c can be written in the form $r^2 + 2rs$ for two different pairs of r and s. For $c \leq 20$, the only possibility is $\boxed{c = 15,}$ corresponding to $r = 1$ and $s = 7$, or $r = 3$ and $s = 1$. □

Problem 8. Suppose that $x > y > 0$ are real numbers satisfying

$$(x-y)^{x^2} = 512$$
$$(x-y)^{xy} = 64$$
$$(x-y)^{y^2} = 16$$

Find $(x+y)^2$.

Solution. We have

$$(x-y)^{x^2-2xy+y^2} = \frac{(x-y)^{x^2} \cdot (x-y)^{y^2}}{((x-y)^{xy})^2} = \frac{2^9 \cdot 2^4}{(2^6)^2} = 2.$$

Squaring both sides gives

$$(x^2 - 2xy + y^2)^{x^2-2xy+y^2} = 4.$$

The only positive number which when raised to itself gives 4 is 2, so we must have $2 = x^2 - 2xy + y^2 = (x-y)^2$. Since $x - y > 0$, we must have $x - y = \sqrt{2}$. Subtituting into the first formula gives $\sqrt{2}^{x^2} = 2^9 = \sqrt{2}^{18}$. This gives $x^2 = 18$, or $x = 3\sqrt{2}$. Similarly, in the third formula we get $\sqrt{2}^{y^2} = 2^4 = \sqrt{2}^8$, so $y^2 = 8$, and $y = 2\sqrt{2}$. Finally,

$$(x+y)^2 = (3\sqrt{2} + 2\sqrt{2})^2 = (5\sqrt{2})^2 = \boxed{50.}$$

□