Math Field Day 2014 Mad Hatter A

Warmup 1.
If $x = 1 + \dfrac{1}{1 + \frac{1}{1+1}}$, evaluate $\dfrac{1}{\frac{1}{x-1} - 1}$.

Warmup 2.
Simplify $\dfrac{2014^2 - 2014 \times 28 + 196}{10^6}$

Warmup 3.
$2^2 \cdot 2^3 \cdot \ \cdots \ \cdot 2^{10} = 4^x$. Evaluate x.

Warmup 4.
Is the sum of 58 consecutive integers
(a) always odd,
(b) always even, or
(c) sometimes odd, sometimes even?

A1
How many real numbers x are there such that $| \, |x| \, - \, 2 \, | \, = \, 2$?

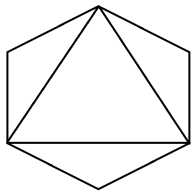
A2
Find the smallest positive integer such that $9! \cdot m$ is a perfect square.

A3
Let $f(x) = \sqrt[3]{1 - x^3}$ for all x.
Solve the equation $f(f(a)) = 2$ for a.

A4
$3 + \log_4(x) = \log_2(x)$.
$x =$?

A5
The roots of the polynomial $x^2 - 41x + c$ differ by 9.
Find c.

A6



Alternate vertices of a regular hexagon are the vertices of an equilateral triangle with sides of length $\sqrt{3}$. How long is a side of the hexagon?

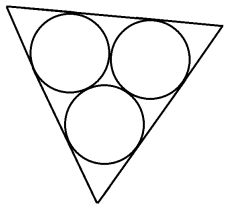
A7
Find the area of the triangle with vertices $(0, 0)$, $(3, 1)$, and $(-1, 3)$.

A8
Find the smallest positive integer x such that $\sin((2013 - x)^\circ) = 0$.

A9
Find the distance between the two points where the line $3y = 4x$ intersects the circle $(x-5)^2 + y^2 = 25$.

A10
$1 + 3 + 5 + \cdots + (2k-1) = 144$. k =???

A11



Three circles of radius 1 inside a triangle are tangent to each other and to the sides of the triangle. How long is a side of the triangle?

A12
If $\log_x(10 - 3x) = 2$, find x.

A13
Find bc if b and c are integers such that $x^3 + 6x^2 + bx + c$ has three consecutive integers as roots.

A14
Simplify $\dfrac{\sin^2(17^\circ)}{\sin^2(34^\circ) + 4\sin^4(17^\circ)}$

A15
To two decimal places, $\log_{10} 6 = 0.78$.
How many digits does 6^{20} have?

A16
Simplify $\dfrac{(2^{2/3} + 2^{1/3})^3 - 6}{2^{2/3} + 2^{1/3}}$

A17
How many times do $y = \tan(x)$ and $y = \cot(x)$ intersect for $0 \le x < 2\pi$, with x in radians?

A18
For $i^2 = -1$, simplify $\left(100 + \dfrac{1}{25}\right)^2 + \left(100i + \dfrac{1}{25i}\right)^2$

A19
If $(2 + a\sqrt{3})(2 + b\sqrt{3}) = 19 + c\sqrt{3}$ for positive integers a, b, and c, find c.

A20 Suppose that 7/15 of mathematics classes have quizzes, 5/6 have graded homework, and 1/10 have neither. What fraction have both?

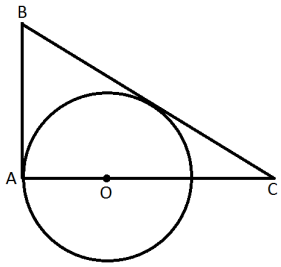
A21
How many arrangements of the letters in the word ELEVEN are there?

A22
Natalie is 2/3 as old as she will be when she is twice as old as she was four years ago. How old is she now?

A23
For what number c does $8y = x^2 + c$ intersect $8x = y^2 + c$ at exactly one point (x, y)?

A24
Given $\dfrac{\log_4 x}{\log_{32} x} = x$, find the value of x.

A25



The circle has center O and is tangent to lines AB and BC. If $OA = 3$ and $AC = 8$, find BC.

A26
How many ways are there to write $81 = abc$ for positive integers a, b, and c such that $a \leq b \leq c$?

A27
Find the largest number r such that $\sin\theta\cos\theta = r$ for some value of θ.

A28
How many arrangements of the letters in MENU have at least one consonant between E and U?

A29
A cube has edges of length 2. Find the distance from the center of one face to a corner of the opposite face.

A30
Three fair six-sided dice are rolled. What is probability that the sum of the numbers that come up is equal to ten?

A31
Suppose that $\log_{64}(4036 - 2x) = \log_9 3$. Find the value of x.

A32
Suppose x is a positive real number, and $(x-2)^{\frac{1}{2}} + (x+2)^{\frac{1}{2}} = (2x)^{\frac{1}{2}}$. $x = ?$

A33
a, b, and c are integers. $x^4 + ax^3 + bx^2 + cx + 9$ has four different rational roots.
a+b+c= ?

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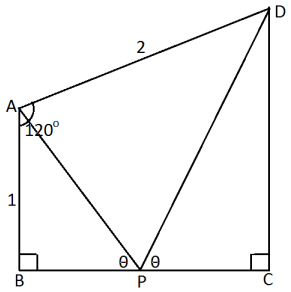

$ABCD$ is a trapezoid . P is a point on side BC.
$\angle DAB = 120^0$, $\angle B = \angle C = 90^o$. $\angle APB = \angle DPC$. $AB = 1$, $AD = 2$.
Find the total length of $AP + PD$.

A36
Evaluate $\cos^4(1^o) - \sin^4(1^o) - \cos(2^o)$.

A37
Find the sum of the roots of $(x-1)^4 - 13(x-1)^2 + 36$.

A38
The maximum value of the function $f(x) = cx - x^2$ is 25. Find c, if $c > 0$.

A39
How many ordered triples (x, y, z) of nonnegative integers are there where $x + y + z = 5$?

A40
Evaluate $f(1) + f(2) + \cdots + f(39)$ for $f(x) = |x - 20| - |20 - x|$.

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Solve the equation $f(f(a)) = 2$ for a.

B4
$a_1 = 1$.
$na_{n+1} = (n+1)a_n \quad$ for $n \geq 1$.
Evaluate a_{2014}.

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The roots of the polynomial $x^2 - 41x + c$ differ by 9.
Find c.

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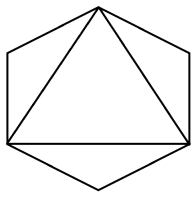

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B7
Find the area of the triangle with vertices $(0, 0)$, $(3, 1)$, and $(-1, 3)$.

B8
Find the largest possible area for a right triangle with hypotenuse of length 10.

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Find the distance between the two points where the line $3y = 4x$ intersects the circle $(x-5)^2 + y^2 = 25$.

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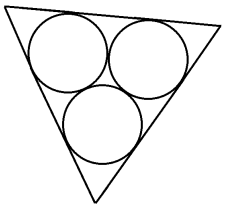

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$a =$???

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Find bc if b and c are integers such that $x^3 + 6x^2 + bx + c$ has three consecutive integers as roots.

B14
Evaluate $f(1) + f(2) + \cdots + f(39)$ for $f(x) = x - 20$.

B15
A trapezoid has vertices $(15, 0)$, $(0, 9)$, $(0, 12)$, and $(c, 0)$, and $c > 15$. Find c.

B16
Simplify $\dfrac{(2^{2/3} + 2^{1/3})^3 - 6}{2^{2/3} + 2^{1/3}}$

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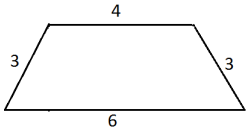

Find the area of the isosceles trapezoid.

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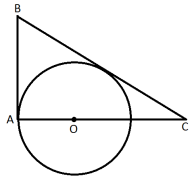
B24
Evaluate $33 + 63 + 93 + \cdots + 333$.

B25



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B27
$$\begin{aligned} x + y &= 3 \\ x^2 + y^2 &= 7 \\ xy &= ? \end{aligned}$$

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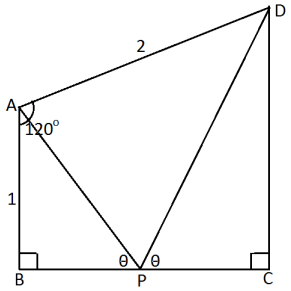

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B36

Every anster is a banster, and every banster is a canster.

Fenchurch may be an anster, a banster, or a canster. In fact, she is exactly two of these things. Which one isn't she?

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Find the sum of the roots of $(x-1)^4 - 13(x-1)^2 + 36$.

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