MATH FIELD DAY 2014

Contestants' Instructions
Team Essay

1. Your team has *forty* minutes to answer this set of questions.

2. All answers must be justified with complete explanations. Your answers should be clear, grammatically correct, and mathematically precise.

3. Your team may turn in at most one answer to each question. Number the answers and submit them in order, starting each problem on a separate page. You are allowed to skip any question and proceed on to later questions. Write on one side of the paper only, and number the pages you turn in consecutively. Do not turn in the problem packets. Write your school name on each page.

4. Your team may be organized in any way you choose. For example, you may designate one team member to write every answer, or you could designate different team members to write different answers. You may also use the chalk board.

5. Only official team members may be in the room during the essay period. Team members may not bring books, calculators, or any other materials into the room.

6. A proctor outside the room will warn you ten minutes before your time is up.

7. Give your answers to the proctor at the end of the period to place in the inner envelope. Please erase all chalk boards in the room and put all scratch paper in the trash can, leaving the room neat.

Crossings

A graph consists of finitely many points in the plane (called vertices) and line segments (called edges) with vertices as their endpoints. We consider a graph to be unchanged when its vertices are moved and the same pairs of vertices remain joined by edges.

We draw a graph by specifying the positions of the vertices in the plane. Any drawing must meet these two conditions: no edge contains any vertex except its two endpoints, and no three edges meet at a common point except a vertex. We can satisfy both conditions by moving vertices slightly from any given positions. Note that edges are straight line segments and not curved arcs.

Throughout the Essay, m and n are positive integers. $K_{m,n}$ is the graph that has $m+n$ vertices marked by m dots and n circles and that has the mn edges joining dots and circles. For example, Figure 1 is a drawing of $K_{3,3}$. It has the $3(3) = 9$ edges joining the three dots on the top of the figure with the three circles on the bottom.

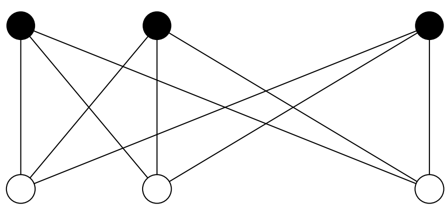

Figure 1

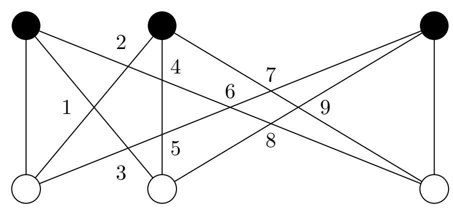

Figure 2

Consider a drawing of a graph. A crossing is a point other than a vertex where two edges intersect. For example, the drawing of $K_{3,3}$ in Figure 1 has nine crossings, which are labeled 1-9 in Figure 2.

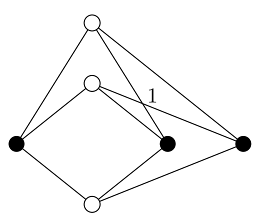

Figure 3

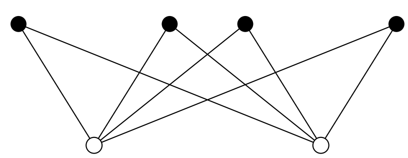

Figure 4

Different drawings of the same graph can have different numbers of crossings. For example, Figure 3 is a drawing of $K_{3,3}$ with just one crossing, which is labeled 1.

Problem 1. Figure 4 is a drawing of $K_{4,2}$. Number the crossings on a copy of the drawing (as in Figure 2).

Problem 2. Draw $K_{4,2}$ twice, once with no crossings and once with one crossing. Label the crossing 1 (as in Figure 3).

Problem 3. Draw $K_{3,3}$ twice, once with three crossings and once with five crossings. Number the crossings in each drawing.

Consider x- and y-axes in the plane, as usual. A drawing of $K_{m,n}$ is balanced if the m dots are points on the x-axis other than the origin, the n circles are points on the y-axis other than the origin, the number of dots on the positive x-axis is equal to or one more than the number of dots on the negative x-axis, and the number of circles on the positive y-axis is either equal to or one more than the number of circles on the negative y-axis. For example, Figure 3 is a balanced drawing of $K_{3,3}$ (where the x- and y-axes aren't shown).

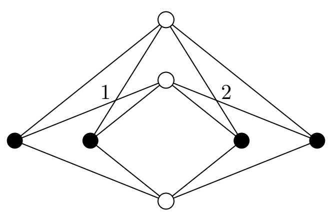

Figure 5

Figure 5 is a balanced drawing of $K_{4,3}$, and its two crossings are numbered.

Problem 4. Give balanced drawings of $K_{5,3}$ and $K_{6,3}$, and number the crossings in each drawing (as in Figures 3 and 5).

Consider any drawing of $K_{3,3}$. Label the dots A, B, C, label the circles P, Q, R, and identify edges by their endpoints. Suppose that none of the six edges AP, PB, BQ, QC, CR, and RA crosses another edge. These six edges form a hexagon, and each of the other three edges AQ, BR, and CP lies either entirely inside of or outside of the hexagon. Then two of the last three edges – say, AQ and BR, by symmetry – either both lie inside the hexagon (Figure 6) or outside (Figure 7). These two edges cross because A and Q alternate with B and R around the hexagon.

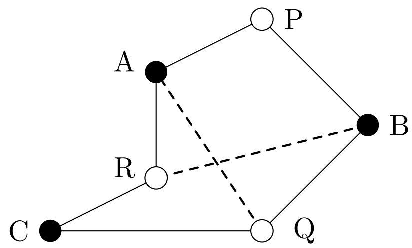

Figure 6

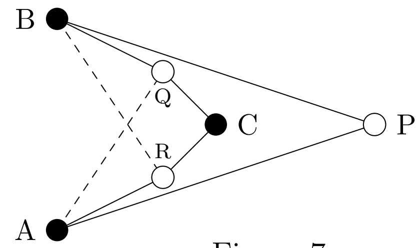

Figure 7

Thus, every drawing of $K_{3,3}$ has at least one crossing. The balanced drawing of $K_{3,3}$ has only one crossing (Figure 3). This illustrates the main result of this Essay: for any positive integer m, no drawing of $K_{m,3}$ or $K_{m,4}$ has fewer crossings than a balanced drawing.

For the rest of the Essay, let t be the integer such that $m = 2t$ or $m = 2t + 1$, depending on whether m is even or odd.

For any positive integer r, a set of r elements has $\frac{r(r-1)}{2}$ subsets of two elements. This is true because there are r choices for one element, $r - 1$ choices for another, and either one of two elements can be picked first. For instance, the set $\{A, B, C, D\}$ with four elements has $\frac{4(3)}{2} = 6$ subsets of two elements:

$$\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \text{ and } \{C, D\}.$$

Consider two dots on the positive x-axis, two circles on the positive y-axis, and the four segments joining dots with circles (Figure 8). Exactly one point in the plane besides the dots and circles lies on two edges.

The two preceding paragraphs may help in doing the next two problems. Problem 5 is illustrated for $t = 2$ by Figure 5 and for $t = 3$ by the second drawing in the answer to Problem 4.

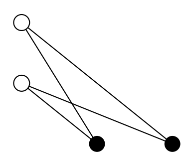
Figure 8

Problem 5. If m is even, explain why a balanced drawing of $K_{m,3}$ has $t^2 - t$ crossings for $m = 2t$.

Problem 6 is illustrated for $t = 1$ by Figure 3 and for $t = 2$ by the first drawing in the answer to Problem 4.

Problem 6. If m is odd, explain why a balanced drawing of $K_{m,3}$ has t^2 crossings for $m = 2t + 1$.

We want to show that every drawing of $K_{m,3}$ has at least as many crossings as a balanced drawing. Consider any drawing of $K_{m,3}$. We associate with it another graph G as follows. G has m vertices, which correspond to the dots in $K_{m,3}$. Two vertices of G are joined by an edge when the corresponding dots do not lie on crossing edges in the drawing of $K_{m,3}$.

For example, consider the drawing of $K_{4,3}$ in Figure 9. The four dots are labeled A through D and the crossings are numbered 1 through 4. Edges through A and C give crossing 1, and edges through B and D give each of the crossings 2 ,3, and 4. The associated graph G has vertices A through D and edges joining all pairs of these vertices except $\{A, C\}$ and $\{B, D\}$. Figure 10 shows G.

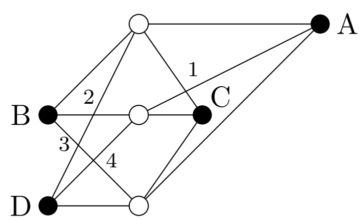

Figure 9

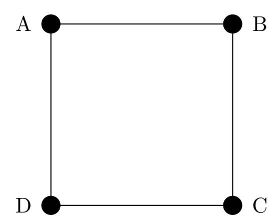

Figure 10

Problem 7. Figure 11 is a drawing of $K_{4,3}$ with dots labeled A through D. Draw the associated graph G. Justify your answer as in the discussion of Figures 9 and 10.

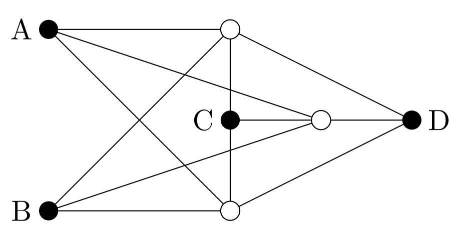

Figure 11

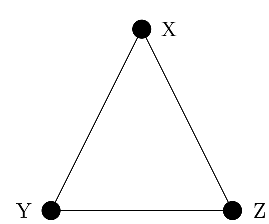

Figure 12

A triangle in a graph G consists of three vertices X, Y, and Z, such that G includes all three of the edges XY, XZ, and YZ (Figure 12). The graph G in Figure 10 has no triangles, and neither does the graph G in the answer to Problem 7. This illustrates the next problem.

Problem 8. Let G be the graph associated with a drawing of $K_{m,3}$. Explain why G has no triangles. Use the observation after Problem 4 that every drawing of $K_{3,3}$ has at least one crossing.

Problem 9. Suppose that a drawing of $K_{m,3}$ has c crossings. Explain why the associated graph G has at least $\frac{1}{2}m^2 - \frac{1}{2}m - c$ edges.

The 2013 Team Essay includes the following result: Let G be a graph with m vertices that has no triangles. If m is even, then G has at most t^2 edges for $m = 2t$. If m is odd, then G has at most $t^2 + t$ edges for $m = 2t + 1$.

Problem 10. Suppose that a drawing of $K_{m,3}$ has c crossings. If m is even, show that $c \geq t^2 - t$ for $m = 2t$. If m is odd, show that $c \geq t^2$ for $m = 2t + 1$. Use Problems 8 and 9 and the result from the 2013 Team Essay.

Problems 5, 6, and 10 show that no drawing of $K_{m,3}$ has fewer crossings than a balanced drawing. Surprisingly, this leads directly to the corresponding result for $K_{m,4}$.

Consider a drawing of $K_{m,4}$. For each vertex X marked with a circle, let c_X be the number of crossings that do not lie on edges through X.

For example, consider the drawing of $K_{3,4}$ in Figure 13. The four vertices marked with circles are labeled P-S, and the crossings are numbered 1-5. We have $c_P = 3$ (since crossings 3, 4, and 5 don't lie on edges through P), $c_Q = 3$ (since crossings 2, 4, and 5 don't lie on edges through Q), $c_R = 3$ (since crossings 1, 2, and 3 don't lie on edges through R, and $c_S = 1$ (since 1 is the only crossing not on an edge through S). Note that

$$c_P + c_Q + c_R + c_S = 3 + 3 + 3 + 1 = 10$$

is twice the total number, 5, of crossings. This illustrates the next problem.

Problem 11. Consider a drawing of $K_{m,4}$, and let P, Q, R, and S be the four vertices marked with circles. Explain why $c_P + c_Q + c_R + c_S$ is twice the number of crossings in the drawing.

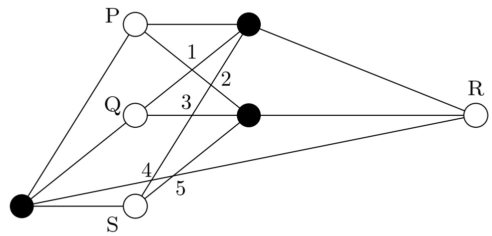

Figure 13

We've seen that every drawing of $K_{m,3}$ has at least as many crossings as a balanced drawing. Combine this result with Problem 11 to do the next problem.

Problem 12. Prove that any drawing of $K_{m,4}$ has at least twice as many crossings as a balanced drawing of $K_{m,3}$.

Problem 13. Why does Problem 12 show that no drawing of $K_{m,4}$ has fewer crossings than a balanced drawing?

Zarankiewicz conjectured in 1954 for all positive integers m and n that no drawing of $K_{m,n}$ has fewer crossings than a balanced drawing. Edges can be curved arcs as well as line segments. The conjecture has been proved for $n \leq 6$ and for $m \leq 10$ and $n \leq 8$. Values of m and n can be interchanged because $K_{m,n} = K_{n,m}$.