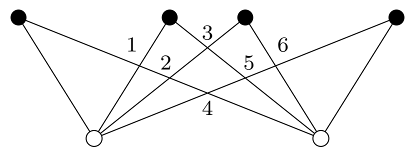
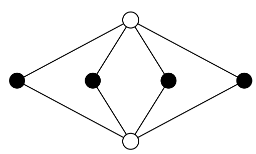
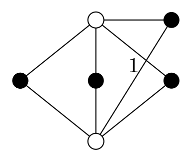
MATH FIELD DAY 2014

Problem 1. Figure 4 is a drawing of $K_{4,2}$. Number the crossings on a copy of the drawing (as in Figure 2).

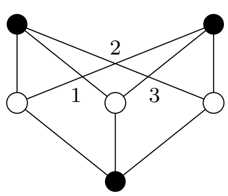


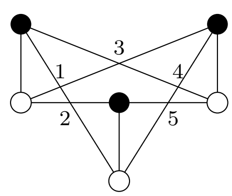
Problem 2. Draw $K_{4,2}$ twice, once with no crossings and once with one crossing. Label the crossing 1 (as in Figure 3).



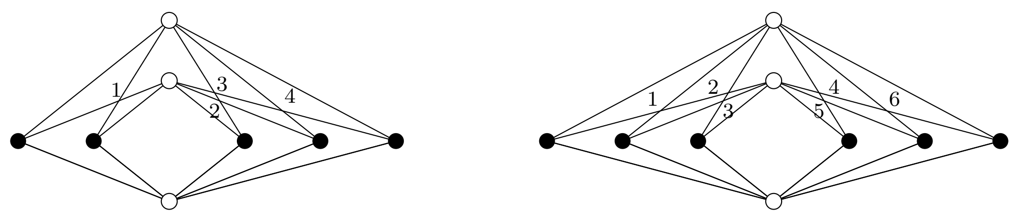


Problem 3. Draw $K_{3,3}$ twice, once with three crossings and once with five crossings. Number the crossings in each drawing.





Problem 4. Give balanced drawings of $K_{5,3}$ and $K_{6,3}$, and number the crossings in each drawing (as in Figures 3 and 5).



Problem 5. If m is even, explain why a balanced drawing of $K_{m,3}$ has $t^2 - t$ crossings for $m = 2t$.

A balanced drawing for $m = 2t$ has t dots on the positive x-axis and t on the negative x-axis. By the discussion before the problem, there are $\frac{t(t-1)}{2}$ pairs of dots on the positive x-axis, and one crossing results form joining each dot in a pair to each of the two circles on the positive y-axis. Each pair of dots gives a different crossing, since no three edges lie on a common point other than a vertex (as the second paragraph of the Essay requires). This gives $\frac{t(t-1)}{2}$ crossings in the first quadrant.

Likewise, the t dots on the negative x-axis gives $\frac{t(t-1)}{2}$ crossings in the second quadrant when they are joined to the two circles on the positive y-axis. No edge through a dot on the positive x-axis crosses an edge through a dot on the negative x-axis, and no edge through the circle on the negative y-axis crosses another edge. Thus, the total number of crossings is

$$\frac{t(t-1)}{2} + \frac{t(t-1)}{2} = t(t-1) = t^2 - t.$$

Problem 6. If m is odd, explain why a balanced drawing of $K_{m,3}$ has t^2 crossings for $m = 2t+1$.

A balanced drawing for $m = 2t+1$ has $t+1$ dots on the positive x-axis and t dots on the negative y-axis. Replacing t with $t+1$ in the first paragraph of the solution to Problem 5 shows that $\frac{(t+1)t}{2}$ crossings result from joining the $t+1$ dots on the positive x-axis to the two circles on the positive y-axis. As in the second paragraph of the solution to Problem 5, $\frac{t(t-1)}{2}$ more crossings result from joining the t dots on the negative x-axis to the two circles on the positive y-axis, and there are no other crossings. Thus, the total number of crossings is

$$\frac{(t+1)t}{2} + \frac{t(t-1)}{2} = \tfrac{1}{2}t^2 + \tfrac{1}{2}t + \tfrac{1}{2}t^2 - \tfrac{1}{2}t = t^2.$$

Problem 7. Figure 11 is a drawing of $K_{4,3}$ with dots labeled A through D. Draw the associated graph G. Justify your answer as in the discussion of Figures 9 and 10.

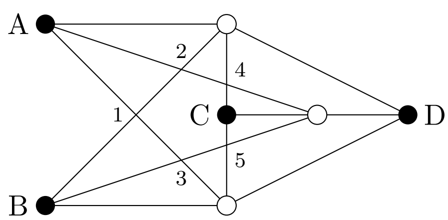


Figure 14

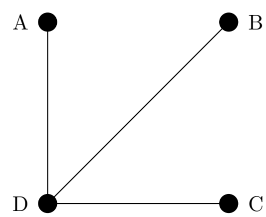


Figure 15

Figure 14 shows Figure 11 with the crossings numbered. Edges through A and B give each of the crossings 1-3, edges through A and C give crossing 4, and edges through B and C give crossing 5. The associated graph G has vertices A through D and edges joining all pairs of these vertices except $\{A, B\}$, $\{A, C\}$, and $\{B, C\}$. Figure 15 shows G.

Problem 8. Let G be the graph associated with a drawing of $K_{m,3}$. Explain why G has no triangles. Use the observation after Problem 4 that every drawing of $K_{3,3}$ has at least one crossing.

Let X, Y, and Z be any three dots in a drawing D of $K_{m,3}$. Removing the other dots and the edges on them from D leaves a drawing E of $K_{3,3}$, since E has three dots X, Y, and Z, the three circles in D, and the nine edges joining dots with circles. E has at least one crossing, by the paragraph after Problem 4. Thus D has a crossing formed by edges on either X and Y, Y and Z, or Z and X. Then the associated graph G is missing at least one of the edges XY, YZ, and ZX. Hence G does not include triangle XYZ for any three vertices X, Y, and Z of G, and so G has no triangles.

Problem 9. Suppose that a drawing of $K_{m,3}$ has c crossings. Explain why the associated graph G has at least $\frac{1}{2}m^2 - \frac{1}{2}m - c$ edges.

Let a drawing D of $K_{m,3}$ have c crossings. Each crossing lies on edges through exactly two dots (by the second condition in the second paragraph of the Essay). However, two dots can lie on edges through more than one crossing (for example, as B and D do in Figure 9). Thus, at most c pairs of dots lie on crossing edges in D. Accordingly, at most c pairs of vertices in G are not joined by edges.

Because D has m dots, G has m vertices, and so it has $\frac{m(m-1)}{2}$ pairs of vertices. At most c of these pairs are not joined by edges (by the previous paragraph), and so at least $\frac{m(m-1)}{2} - c$ pairs of vertices in G are joined by edges. In other words, G has at least $\frac{1}{2}m^2 - \frac{1}{2}m - c$ edges.

Problem 10. Suppose that a drawing of $K_{m,3}$ has c crossings. If m is even, show that $c \geq t^2 - t$ for $m = 2t$. If m is odd, show that $c \geq t^2$ for $m = 2t + 1$. Use Problems 8 and 9 and the result from the 2013 Team Essay.

First let m be even. G has m vertices (since $K_{m,3}$ has m dots), and it has no triangles (by Problem 8). Then G has at most t^2 edges for $m = 2t$ (by the 2013 Team Essay). Since G has at least $\frac{1}{2}m^2 - \frac{1}{2}m - c$ edges (by Problem 9), it follows that

$$\begin{aligned} t^2 &\geq \tfrac{1}{2}m^2 - \tfrac{1}{2}m - c \\ &= \tfrac{1}{2}(2t)^2 - \tfrac{1}{2}(2t) - c \\ &= 2t^2 - t - c, \end{aligned}$$

and so $c \geq t^2 - t$.

Next, let m be odd. G has m vertices (since $K_{m,3}$ has m dots), and it has no triangles (by Problem 8). Then G has at most $t^2 + t$ edges for $m = 2t + 1$ (by the 2013 Team Essay). Since G has at least $\frac{1}{2}m^2 - \frac{1}{2}m - c$ edges (by Problem 9), it follows that

$$\begin{aligned} t^2 + t &\geq \tfrac{1}{2}m^2 - \tfrac{1}{2}m - c \\ &= \tfrac{1}{2}(2t+1)^2 - \tfrac{1}{2}(2t+1) - c \\ &= \tfrac{1}{2}(4t^2 + 4t + 1) - t - \tfrac{1}{2} - c \\ &= 2t^2 + 2t + \tfrac{1}{2} - t - \tfrac{1}{2} - c \\ &= 2t^2 + t - c, \end{aligned}$$

and so $c \geq t^2$.

Problem 11. Consider a drawing of $K_{m,4}$, and let P, Q, R, and S be the four vertices marked with circles. Explain why $c_P + c_Q + c_R + c_S$ is twice the number of crossings in the drawing.

Every crossing is the intersection of edges through two circles. By symmetry, it suffices to show that every crossing of edges through the circles P and Q is counted twice in the sum

$$c_P + c_Q + c_R + c_S. \tag{1}$$

Let edges through P and Q cross at a point V. Neither c_P nor c_Q counts the crossing but both c_R and c_S do (since P and Q are the only two circles on edges through V, by the second condition in the second paragraph of the Essay). Thus, the sum (1) counts the crossing at V exactly twice, as desired.

Problem 12. Prove that any drawing of $K_{m,4}$ has at least twice as many crossings as a balanced drawing of $K_{m,3}$.

Let D be a drawing of $K_{m,4}$. Let t be the number of crossings in D, and let u be the number of crossings in a balanced drawing of $K_{m,3}$. We must show that $t \geq 2u$.

Let P-S be the four vertices marked as circles in D, and let X be any one of these vertices. Dropping X and the edges on it from D gives a drawing of $K_{m,3}$ (since the remaining edges join the m dots with the three remaining circles). Since this drawing has c_X crossings, and since every drawing of $K_{m,3}$ has at least as many crossings as a balanced drawing, we have $c_X \geq u$. Then the sum (1) is at least $4u$. Because the sum (1) is $2t$ (by Problem 11), we have $2t \geq 4u$. Dividing by 2 gives $t \geq 2u$, as desired.

Problem 13. Why does Problem 12 show that no drawing of $K_{m,4}$ has fewer crossings than a balanced drawing?

Let E be a balanced drawing of $K_{m,4}$, and let F be a balanced drawing of $K_{m,3}$. E and F have the same number of crossings formed by edges through their two circles on the positive y-axis, and these are all the crossings in F (by the solutions to Problems 5 and 6). E also has this number of crossings formed by edges through its two circles on the negative y-axis. Dots are the only points where an edge of E through a circle on the positive y-axis intersects an edge through a circle on the negative y-axis. Thus, E has twice as many crossings as F.

Because any drawing of $K_{m,4}$ has at least twice as many crossings as F (by Problem 12), it has at least as many as E (by the previous paragraph). In other words, no drawing of $K_{m,4}$ has fewer crossings than a balanced drawing.