

## APPENDIX 3A: Equity Valuation

The valuation process for an equity instrument (such as common stock or a share) involves finding the present value of an infinite series of cash flows on the equity discounted at an appropriate interest rate. Cash flows from holding equity come from dividends paid out by the firm over the life of the stock, which in expectation can be viewed as infinite since a firm (and thus the dividends it pays) has no defined maturity or life. Even if an equity holder decides not to hold the stock forever, he or she can sell it to someone else who in a fair and efficient market is willing to pay the present value of the remaining (expected) dividends to the seller at the time of sale. Dividends on equity are that portion of a firm's earnings paid out to the stockholders. Those earnings retained are normally reinvested to produce future income and future dividends for the firm and its stockholders. Thus, conceptually, the fair price paid for investing in stocks is the present value of its current and future dividends. Growth in dividends occurs primarily because of growth in the firm's earnings, which is, in turn, a function of the profitability of the firm's investments and the percentage of these profits paid out as dividends rather than being reinvested in the firm. Thus, earnings growth, dividend growth, and stock value (price) will generally be highly correlated.

We begin by defining the variables we will use to value an equity:

$D_t$  = Dividend paid out to stockholders at the end of the year  $t$

$P_t$  = Price of a firm's common stock at the end of the year  $t$

$P_0$  = Current price of a firm's common stock

$i_s$  = Interest rate used to discount cash flows on an investment in a stock

As described above, time value of money equations can be used to evaluate a stock from several different perspectives. For example, the realized rate of return ( $rr$ ) is the appropriate interest rate (discount rate) to apply to cash flows when evaluating the historical performance of an equity.

### EXAMPLE 3-10 Calculation of Realized Rate of Return on a Stock Investment

Suppose you owned a stock for the last two years. You originally bought the stock two years ago for \$25 ( $P_0$ ) and just sold it for \$35 ( $P_2$ ). The stock paid an annual dividend of \$1 on the last day of each of the past two years. Your realized rate of return on the stock investment can be calculated using the following time value of money equation:

$$P_0 = D(PVIFA_{i_s,2}) + P_2(PVIF_{i_s,2})$$

or

$$25 = 1(PVIFA_{i_s,2}) + 35(PVIF_{i_s,2})$$

Solving for  $i_s$ , your annual realized rate of return ( $rr$ ) on this investment was  $i_s = rr = 22.02$  percent.

$$25 = 1(1.4912) + 35(0.6716)$$

The expected rate of return ( $Err$ ) is the appropriate interest rate when analyzing the expected future return on stocks, assuming the investor buys the stock at its current market price, receives all promised payments, and sells the stock at the end of his or her investment horizon.

**EXAMPLE 3-11** Calculation of Expected Rate of Return on a Stock Investment

You are considering the purchase of a stock that you expect to own for the next three years. The current market price of the stock is \$32 ( $P_0$ ) and you expect to sell it for \$45 in three years time ( $\tilde{P}_3$ ). You also expect the stock to pay an annual dividend ( $\tilde{D}$ ) of \$1.50 on the last day of each of the next three years. Your expected return on the stock investment can be calculated using the following time value of money equation:

$$P_0 = \tilde{D} (PVIFA_{i_s,3}) + \tilde{P}_3(PVIF_{i_s,3})$$

or

$$32 = 1.50 (PVIFA_{i_s,3}) + 45 (PVIF_{i_s,3})$$

Solving for  $i_s$ , your annual expected rate of return ( $Err$ ) on this investment is  $i_s = Err = 16.25$  percent.

Finally, the required rate of return ( $rrr$ ) is the appropriate interest rate when analyzing the fair value of a stock investment over its whole lifetime. The fair value of a stock reflects the present value of all relevant (but uncertain) cash flows to be received by an investor discounted at the required rate of return ( $rrr$ )—the interest rate or return that should be earned on the investment given its risk. Present value methodology applies time value of money to evaluate a stock's cash flows over its life as follows:

$$P_0 = \frac{\tilde{D}_1}{(1+i_s)^1} + \frac{\tilde{D}_2}{(1+i_s)^2} + \dots + \frac{\tilde{D}_\infty}{(1+i_s)^\infty}$$

The price or value of a stock is equal to the present value of its future dividends ( $\tilde{D}_t$ ), whose values are uncertain. This requires an infinite number of future dividend values to be estimated, which makes the equation above difficult to use for stock valuation and  $rrr$  calculation in practice. Accordingly, assumptions are normally made regarding the expected pattern of the uncertain flow of dividends over the life of the stock. Two assumptions that are commonly used are (1) zero growth in dividends over the (infinite) life of the stock; and (2) a constant growth rate in dividends over the (infinite) life of the stock.

**Zero Growth in Dividends**

Zero growth in dividends means that dividends on a stock are expected to remain at a constant level forever. Thus,  $D_0 = D_1 = D_2 = \dots = D_\infty = D$ . Accordingly, the equity valuation formula can be written as follows:

$$P_0 = \frac{\tilde{D}_1}{(1+i_s)^1} + \frac{\tilde{D}_2}{(1+i_s)^2} + \dots + \frac{\tilde{D}_\infty}{(1+i_s)^\infty} = D \sum_{t=1}^{\infty} \left( \frac{1}{1+i_s} \right)^t$$

where

$D$  = Current (time 0) value of dividends

$D_t$  = Value of dividends at time  $t = 1, 2, \dots, \infty$

or<sup>20</sup>

$$P_0 = D/i_s$$

20. Remember that, in the limit:

$$\sum_{t=1}^{\infty} \left( \frac{1}{1+x} \right)^t = \left( \frac{1}{1+x} \right)^1 + \left( \frac{1}{1+x} \right)^2 + \dots + \left( \frac{1}{1+x} \right)^\infty = \frac{1}{x}$$

Thus:

$$\sum_{t=1}^{\infty} \left( \frac{1}{1+i_s} \right)^t = \frac{1}{i_s}$$

This formula can be generalized as follows:

$$P_t = D/i_s$$

The value of a stock with zero growth in dividends is equal to the (current) dividend divided by the return on the stock. If the required rate of return ( $rrr$ ) is applied to the formula ( $i_s = rrr$ ), the price we solve for is the fair market price. If the expected return ( $Err$ ) is applied to the formula ( $i_s = Err$ ), the price we solve for is the current market price. Furthermore, the formula can be rearranged to determine a return on the stock if it were purchased at a price,  $P_0$ .<sup>21</sup>

$$i_s = D/P_0$$

If the fair market price is applied to this formula, the return we solve for is the required rate of return ( $rrr$ ). If the current market price is applied to the formula, the price we solve for is the expected return ( $Err$ ). Recall from above, in efficient markets the required rate of return equals the expected rate of return and thus the current market price on a security equals its fair market value.

### EXAMPLE 3-12 Calculation of Stock Price with Zero Growth in Dividends

A stock you are evaluating is expected to pay a constant dividend of \$5 per year each year into the future. The expected rate of return ( $Err$ ) on the stock is 12 percent. The current market value (or price) of this stock is calculated as follows:

$$P_0 = 5/0.12 = \$41.67$$

### Constant Growth in Dividends

Constant growth in dividends means that dividends on a stock are expected to grow at a constant rate,  $g$ , each year into the future. Thus,  $D_1 = D_0(1 + g)^1$ ,  $D_2 = D_0(1 + g)^2$ , ...,  $D_\infty = D_0(1 + g)^\infty$ . Accordingly, the equity valuation formula can now be written as follows:

$$P_0 = \frac{D_0(1 + g)^1}{(1 + i_s)^1} + \frac{D_0(1 + g)^2}{(1 + i_s)^2} + \dots + \frac{D_0(1 + g)^\infty}{(1 + i_s)^\infty} = D_0 \sum_{t=1}^{\infty} \left( \frac{1 + g}{1 + i_s} \right)^t$$

or<sup>22</sup>

$$P_0 = \frac{D_0(1 + g)^1}{i_s - g} = \frac{D_1}{i_s - g}$$

This formula can be generalized as follows:

$$P_t = \frac{D_0(1 + g)^t}{i_s - g} = \frac{D_{t+1}}{i_s - g}$$

If the required rate of return ( $rrr$ ) is applied to the formula ( $i_s = rrr$ ), the price we solve for is the fair market price. If the expected return ( $Err$ ) is applied to the formula ( $i_s = Err$ ), the price we solve for is the current market price. The equity valuation formula can also be rearranged to determine a rate of return on the stock if it were purchased at a price  $P_0$ :

$$i_s = \frac{D_0(1 + g)}{P_0} + g = \frac{D_1}{P_0} + g$$

21. This is also referred to as the current dividend yield on a stock.

22. Remember that in the limit:

$$\sum_{t=1}^{\infty} \left( \frac{1 + g}{1 + i_s} \right)^t = \sum_{t=1}^{\infty} \left( \frac{1}{1 + \frac{i_s - g}{1 + g}} \right)^t = \frac{1 + g}{i_s - g}$$

If the fair market price is applied to the formula, the return we solve for is the required rate of return ( $rrr$ ). If the current market price is applied to the formula, the price we solve for is the expected return ( $Err$ ).

### EXAMPLE 3-13 Calculation of Stock Price with Constant Growth in Dividends

A stock you are evaluating paid a dividend at the end of last year of \$3.50. Dividends have grown at a constant rate of 2 percent per year over the last 20 years, and this constant growth rate is expected to continue into the future. The required rate of return ( $rrr$ ) on the stock is 10 percent. The fair present value (or price) of this stock is calculated as follows:

$$P_0 = \frac{3.50(1 + .02)}{.10 - .02} = \$44.625$$

The investor would be willing to pay no more than \$44.625 for this stock.

### EXAMPLE 3-14 Calculation of the Expected Rate of Return ( $Err$ ) on a Stock with Constant Growth in Dividends

A stock you are evaluating paid a dividend at the end of last year of \$4.80. Dividends have grown at a constant rate of 1.75 percent per year over the last 15 years, and this constant growth rate is expected to continue into the future. The stock is currently selling at a price of \$52 per share. The expected rate of return on this stock is calculated as follows:

$$i_s = \frac{4.80(1 + .0175)}{52} + .0175 = 11.14\%$$

### Supernormal (or Nonconstant) Growth in Dividends

Firms often experience periods of supernormal or nonconstant dividend growth, after which dividend growth settles at some constant rate. The stock value for a firm experiencing supernormal growth in dividends is, like firms with zero or constant dividend growth, equal to the present value of the firm's expected future dividends. However, in this case, dividends during the period of supernormal (nonconstant) growth must be evaluated individually. The constant growth in dividends model can then be adapted to find the present value of dividends following the supernormal growth period.

To find the present value of a stock experiencing supernormal or nonconstant dividend growth, we calculate the present value of dividends during the two different growth periods. A three-step process is used as follows:

- Step 1:** Find the present value of the dividends during the period of supernormal growth;
- Step 2:** Find the price of the stock at the end of the supernormal growth period (when constant growth in dividends begins) using the constant growth in dividends model. Then discount this price to a present value.
- Step 3:** Add the two components of the stock price together.

### EXAMPLE 3-15 Calculation of Stock Price with Supernormal or Nonconstant Growth in Dividends

A stock you are evaluating is expected to experience supernormal growth in dividends of 10 percent,  $g_s$ , over the next five years. Following this period, dividends are expected to grow at a constant rate of 4 percent,  $g$ . The stock paid a dividend of \$4 last year, and the required

rate of return on the stock is 15 percent. The fair present value of the stock is calculated as follows:

**Step 1:** Find the present value of the dividends during the period of supernormal growth.

Year	Dividends ( $D_0(1 + g_s)^t$ )	PVIF <sub>15%,t</sub>	Present Value
1	$4(1 + .1)^1 = 4.400$	.8696	3.826
2	$4(1 + .1)^2 = 4.840$	.7561	3.659
3	$4(1 + .1)^3 = 5.324$	.6575	3.500
4	$4(1 + .1)^4 = 5.856$	.5718	3.349
5	$4(1 + .1)^5 = 6.442$	.4972	3.203
Present value of dividends during supernormal growth period			\$17.537

**Step 2:** Find present value of dividends after period of supernormal growth.

a. Find stock value at beginning of constant growth period:

$$P_5 = \frac{D_6}{k_s - g} = \frac{D_0(1 + g_s)^5(1 + g)^1}{k_s - g} = \frac{4(1 + .1)^5(1 + .04)^1}{.15 - .04} = \$60.906$$

b. Find present value of constant growth dividends:

$$P_0 = P_5(PVIF_{15\%,5}) = 60.906(.4972) = \$30.283$$

**Step 3:** Find present value of stock = Value during supernormal growth period + Value during normal growth period.

$$\$17.537 + \$30.283 = \$47.820$$

## APPENDIX 3B: Duration and Immunization

In the body of the chapter, you learned how to calculate duration and came to understand that the duration measure has economic meaning because it indicates the interest sensitivity or elasticity of an asset's or liability's value. For FIs, the major relevance of duration is as a measure for managing interest rate risk exposure. Also important is duration's role in allowing an FI to hedge or immunize its balance sheet or some subset on that balance sheet against interest rate risk. The following sections consider two examples of an FI's use of the duration measure for immunization purposes. The first is its use by insurance company and pension fund managers to help meet promised cash flow payments to policyholders or beneficiaries at a particular time in the future. The second is its use in immunizing or insulating an FI's balance sheet against interest rate risk.

### Duration and Immunizing Future Payments

Frequently, pension fund and life insurance company managers face the problem of structuring their asset investments so they can pay a given cash amount to policyholders in some future period. The classic example of this is an insurance policy that pays the holder some lump sum when the holder reaches retirement age. The risk to the life insurance company manager is that interest rates on the funds generated from investing the holder's premiums could fall. Thus, the accumulated returns on the premiums invested could not meet the target or promised amount. In effect, the insurance company would be forced to draw down its reserves and net worth to meet its payout commitments. (See Chapter 15 for a discussion of this risk.)

Suppose that it is 2004 and the insurer must make a guaranteed payment to an investor in five years, 2009. For simplicity, we assume that this target guaranteed payment is \$1,469, a lump-sum policy payout on retirement, equivalent to investing \$1,000 at an annually

compounded rate of 8 percent over five years. Of course, realistically, this payment would be much larger, but the underlying principles of the example do not change by scaling up or down the payout amount.

To immunize or protect itself against interest rate risk, the insurer needs to determine which investments would produce a cash flow of exactly \$1,469 in five years, regardless of what happens to interest rates in the immediate future. By investing either in a five-year maturity and duration zero-coupon bond or a coupon bond with a five-year duration, the FI would produce a \$1,469 cash flow in five years, no matter what happens to interest rates in the immediate future. Next we consider the two strategies: buying five-year deep-discount bonds and buying five-year duration coupon bonds.

**Buy Five-Year Deep-Discount Bonds.** Given a \$1,000 face value and an 8 percent yield and assuming annual compounding, the current price per five-year discount bond is \$680.58 per bond

$$P = 680.58 = \frac{1,000}{(1.08)^5}$$

If the insurer buys 1.469 of these bonds at a total cost of \$1,000 in 2004, these investments would produce \$1,469 on maturity in five years. The reason is that the duration of this bond portfolio exactly matches the target horizon for the insurer's future liability to its policyholders. Intuitively, since the issuer of the zero-coupon discount bonds pays no intervening cash flows or coupons, future changes in interest rates have no reinvestment income effect. Thus, the return would be unaffected by intervening interest rate changes.

**Buy A Five-Year Duration Coupon Bond.** Suppose that no five-year discount bonds exist. In this case, the portfolio manager may seek to invest in appropriate duration coupon bonds to hedge interest rate risk. In this example, the appropriate investment is in five-year duration coupon-bearing bonds. Consider a six-year maturity bond with an 8 percent coupon paid annually, an 8 percent yield and \$1,000 face value. The duration of this six-year maturity bond is computed as 4.993 years, or approximately 5 years (Table 3–14). By buying this six-year maturity, five-year duration bond in 2004 and holding it for five years until 2009, the term exactly matches the insurer's target horizon. We show in the next set of examples that the cash flow generated at the end of five years is \$1,469 whether interest rates stay at 8 percent or instantaneously (immediately) rise to 9 percent or fall to 7 percent. Thus, buying a coupon bond whose duration exactly matches the investment time horizon of the insurer also immunizes the insurer against interest rate changes.

**TABLE 3-14** The Duration of a Six-Year Bond with 8 Percent Coupon Paid Annually and an 8 Percent Yield

$t$	$CF_t$	$\frac{1}{(1 + 8\%)^t}$	$\frac{CF_t}{(1 + 8\%)^t}$	$\frac{CF_t * t}{(1 + 8\%)^t}$
1	80	0.9259	74.07	74.07
2	80	0.8573	68.59	137.18
3	80	0.7938	63.51	190.53
4	80	0.7350	58.80	235.20
5	80	0.6806	54.45	272.25
6	1,080	0.6302	680.58	4,083.48
			1,000.00	4,992.71

$$D = 4,992.71/1,000.00 = 4.993 \text{ years}$$

**EXAMPLE 3-16 Interest Rates Remain at 8 Percent**

The cash flows received by the insurer on the bond if interest rates stay at 8 percent throughout the five years are as follows:

1. Coupons, $5 \times \$80$	\$ 400
2. Reinvestment income	69
3. Proceeds from sale of bond at end of the fifth year	<u>1,000</u>
	\$1,469

We calculate each of the three components of the insurer's income from the bond investment as follows:

- Coupons.** The \$400 from coupons is simply the annual coupon of \$80 received in each of the five years.
- Reinvestment income.** Because the coupons are received annually, they can be reinvested at 8 percent as they are received, generating an additional cash flow of \$69. To understand this, consider the coupon payments as an annuity stream of \$80 invested at 8 percent at the end of each year for five years. The future value of the annuity stream is calculated using a future value of an annuity factor (*FVIFA*) as  $\$80 (FVIFA_{5 \text{ years}, 8\%}) = 80 (5.867) = \$469$ . Subtracting the \$400 of invested coupon payments leaves \$69 of reinvestment income.
- Bond sale proceeds.** The proceeds from the sale are calculated by recognizing that the six-year bond has just one year left to maturity when the insurance company sells it at the end of the fifth year (i.e., year 2009). That is:

↓ Sell	<b>\$1,080</b>
Year 5 (2008)	Year 6 (2009)

What fair market price can the insurer expect to receive upon selling the bond at the end of the fifth year with one year left to maturity? A buyer would be willing to pay the present value of the \$1,080—final coupon plus face value—to be received at the end of the one remaining year, or:

$$P_5 = \frac{1.080}{1.08} = \$1.000$$

Thus the insurer would be able to sell the one remaining cash flow of \$1,080, to be received in the bond's final year, for \$1,000.

Next we show that since this bond has a duration of five years, matching the insurer's target period, even if interest rates were to instantaneously fall to 7 percent or rise to 9 percent, the expected cash flows from the bond still would sum exactly to \$1,469. That is, the coupons plus reinvestment income plus principal received at the end of the fifth year would be immunized. In other words, the cash flows on the bond are protected against interest rate changes.

**EXAMPLE 3-17 Interest Rates Fall to 7 Percent**

In this example with falling interest rates, the cash flows over the five years are as follows:

1. Coupons, $5 \times \$80$	\$ 400
2. Reinvestment income	60
3. Bond sale proceeds	<u>1,009</u>
	\$1,469

Thus, the amount of the total proceeds over the five years is unchanged from proceeds generated when interest rates were 8 percent. To see why this occurs, consider what happens to the three parts of the cash flow when rates fall to 7 percent:

1. *Coupons.* These are unchanged, since the insurer still receives five annual coupons of \$80 (\$400).
2. *Reinvestment income.* The coupons can now be reinvested only at the lower rate of 7 percent. Thus, at the end of five years \$80 ( $FVIFA_{5 \text{ years}, 7\%} = 80(5.751) = \$460$ ). Subtracting the \$400 in original coupon payments leaves \$60. Because interest rates have fallen, the investor has \$9 less in reinvestment income at the end of the five-year planning horizon.
3. *Bond sale proceeds.* When the six-year maturity bond is sold at the end of the fifth year with one cash flow of \$1,080 remaining, investors would be willing to pay more:

$$P_5 = \frac{1,080}{1.07} = \$1,009$$

That is, the bond can be sold for \$9 more than when rates were 8 percent. The reason is that investors can get only 7 percent on newly issued bonds, but this older bond was issued with a higher coupon of 8 percent.

A comparison of reinvestment income with bond sale proceeds indicates that the decrease in rates has produced a *gain* of \$9 on the bond sale proceeds. This offsets the loss of reinvestment income of \$9 as a result of reinvesting at a lower interest rate. Thus, total cash flows remain unchanged at \$1,469.

### EXAMPLE 3-18 Interest Rates Rise to 9 Percent

In this example with rising interest rates, the proceeds from the bond investment are as follows:

1. Coupons, $5 \times \$80$	\$ 400
2. Reinvestment income [ $(FVIFA_{5 \text{ years}, 9\%})80 - 400$ ]	78 [ $(FVIFA_{5 \text{ years}, 9\%}) = 5.985$ ]
3. Bond sale proceeds (1,080/1.09)	991
	<u>\$1,469</u>

Notice that the rise in interest rates from 8 to 9 percent leaves the final terminal cash flow unaffected at \$1,469. The rise in rates has generated \$9 extra reinvestment income (\$78 – \$69), but the price at which the bond can be sold at the end of the fifth year has declined from \$1,000 to \$991, equal to a capital loss of \$9. Thus, the gain in reinvestment income is exactly offset by the capital loss on the sale of the bond.

The preceding examples demonstrate that matching the duration of a coupon bond to the FI's target or investment horizon *immunizes* it against instantaneous shocks to interest rates. The gains or losses on reinvestment income that result from an interest rate change are exactly offset by losses or gains from the bond proceeds on sale.

## APPENDIX 3C: More on Convexity

In the main text of this chapter, we explained why convexity is a desirable feature for assets. In this appendix we then ask: Can we measure convexity? And can we incorporate this measurement in the duration model to adjust for or offset the error in prediction due to its presence? The answer to both questions is yes.

Theoretically speaking, duration is the slope of the price–yield curve, and convexity, or curvature, is the change in the slope of the price–yield curve. Consider the total effect of a change in interest rates on a bond's price as being broken into a number of separate effects. The precise mathematical derivation of these separate effects is based on a Taylor series expansion that you might remember from your math classes. Essentially, the first-order effect ( $dP/dR$ ) of an interest rate change on the bond's price is the price–yield curve slope effect, which is measured by duration. The second-order effect ( $d^2P/d^2R$ ) measures the change in the slope of the price–yield curve; this is the curvature or convexity effect. There are also third-, fourth-, and higher-order effects from the Taylor series expansion, but for all practical purposes these effects can be ignored.

We have noted that overlooking the curvature of the price–yield curve may cause errors in predicting the interest sensitivity of a portfolio of assets and liabilities, especially when yields change by large amounts. We can adjust for this by explicitly recognizing the second-order effect of yield changes by measuring the change in the slope of the price–yield curve around a given point. Just as  $D$  (duration) measures the slope effect ( $dP/dR$ ), we introduce a new parameter ( $CX$ ) to measure the curvature effect ( $d^2P/d^2R$ ) of the price–yield curve.

The resulting equation, predicting the change in a security's price ( $\Delta P/P$ ), is:

$$\frac{\Delta P}{P} = -D \frac{\Delta R}{(1 + R)} + \frac{1}{2} CX(\Delta R)^2 \quad (1)$$

or:

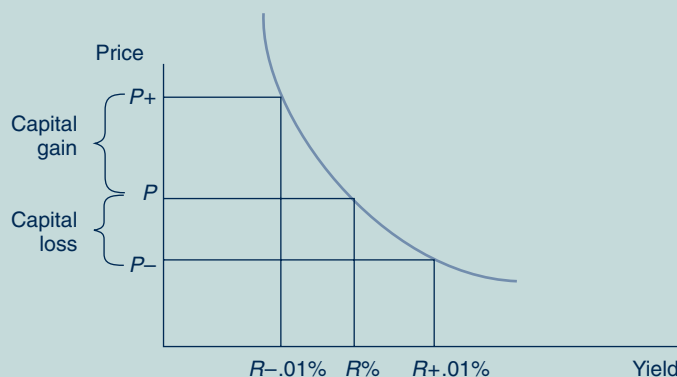
$$\frac{\Delta P}{P} = -MD\Delta R + \frac{1}{2} CX(\Delta R)^2 \quad (2)$$

The first term in Equation 1 is the simple duration model that over- or underpredicts price changes for large changes in interest rates, and the second term is the second-order effect of interest rate changes, that is, the convexity or curvature adjustment. In Equation 1, the first term  $D$  can be divided by  $1 + R$  to produce what we called earlier *modified duration* ( $MD$ ). You can see this in Equation 2. This form is more intuitive because we multiply  $MD$  by the simple change in  $R$  ( $\Delta R$ ) rather than by the discounted change in  $R$  ( $\Delta R/(1 + R)$ ). In the convexity term, the number  $1/2$  and  $(\Delta R)^2$  result from the fact that the convexity effect is the second-order effect of interest rate changes while duration is the first-order effect. The parameter  $CX$  reflects the degree of curvature in the price–yield curve at the current yield level, that is, the degree to which the *capital gain effect* exceeds the *capital loss effect* for an equal change in yields up or down. At best, the FI manager can only approximate the curvature effect by using a parametric measure of  $CX$ . Even though calculus is based on infinitesimally small changes, in financial markets the smallest change in yields normally observed is one basis point, or a  $1/100$  of 1 percent change. One possible way to measure  $CX$  is introduced next.

As just discussed, the convexity effect is the degree to which the capital gain effect more than offsets the capital loss effect for an equal increase and decrease in interest rates at the current interest rate level. In Figure 3–10 we depict yields changing upward by one basis point ( $R + .01\%$ ) and downward by one basis point ( $R - .01\%$ ). Because convexity measures the curvature of the price–yield curve around the rate level  $R$  percent, it intuitively measures the degree to which the capital gain effect of a small yield decrease exceeds the capital loss effect of a small yield increase.<sup>23</sup> By definition, the  $CX$  parameter equals:

$$CX = \text{Scaling factor} \left[ \begin{array}{l} \text{The capital loss from a one-basis-point rise in yield} \\ \text{(negative effect)} \end{array} + \begin{array}{l} \text{The capital gain from a one-basis-point fall in yield} \\ \text{(positive effect)} \end{array} \right]$$

23. We are trying to approximate as best we can the change in the slope of the price–yield curve at  $R$  percent. In theory, the changes are infinitesimally small ( $dR$ ), but in reality, the smallest yield change normally observed is one basis point ( $\Delta R$ ).

**FIGURE 3-10 Convexity and the Price-Yield Curve**

The sum of the two terms in the brackets reflects the degree to which the capital gain effect exceeds the capital loss effect for a small one-basis-point interest rate change down and up. The scaling factor normalizes this measure to account for a larger 1 percent change in rates. Remember, when interest rates change by a large amount, the convexity effect is important to measure. A commonly used scaling factor is  $10^8$  so that:<sup>24</sup>

$$CX = 10^8 \left[ \frac{\Delta P_-}{P} + \frac{\Delta P_+}{P} \right]$$

*Calculation of CX.* To calculate the convexity of the 8 percent coupon, 8 percent yield, six-year maturity Eurobond that had a price of \$1,000:<sup>25</sup>

$$CX = 10^8 \left[ \frac{999.53785 - 1,000}{1,000} + \frac{1,000.46243 - 1,000}{1,000} \right]$$

Capital loss from      Capital gain from  
a one-basis-point      a one-basis-point  
increase in rates      decrease in rates

$$CX = 10^8 [0.00000028]$$

$$CX = 28$$

This value for  $CX$  can be inserted into the bond price prediction Equation 2 with the convexity adjustment:

$$\frac{\Delta P}{P} = -MD \Delta R + \frac{1}{2}(28)\Delta R^2$$

Assuming a 2 percent increase in  $R$  (from 8 to 10 percent):

$$\begin{aligned} \frac{\Delta P}{P} &= - \left[ \frac{4.993}{1.08} \right] .02 + \frac{1}{2} (28) (.02)^2 \\ &= -.0925 + .0056 = -.0869 \text{ or } -8.69\% \end{aligned}$$

24. This is consistent with the effect of a 1 percent (100 basis points) change in rates.

25. You can easily check that \$999.53785 is the price of the six-year bond when rates are 8.01 percent and \$1,000.46243 is the price of the bond when rates fall to 7.99 percent. Since we are dealing in small numbers and convexity is sensitive to the number of decimal places assumed, use at least five decimal places in calculating the capital gain or loss, in fact, the more decimal places used, the greater the accuracy of the  $CX$  measure.

TABLE 3-15 Properties of Convexity

1. Convexity Increases with Bond Maturity			2. Convexity-Varies with Coupon		3. For Same Duration, Zero-Coupon Bonds Are Less Convex than Coupon Bonds	
Example			Example		Example	
A	B	C	A	B	A	B
$N = 6$	$N = 18$	$N = \infty$	$N = 6$	$N = 6$	$N = 6$	$N = 5$
$R = 8\%$	$R = 8\%$	$R = 8\%$	$R = 8\%$	$R = 8\%$	$R = 8\%$	$R = 8\%$
$C = 8\%$	$C = 8\%$	$C = 8\%$	$C = 8\%$	$C = 0\%$	$C = 8\%$	$C = 0\%$
$D = 5$	$D = 10.12$	$D = 13.5$	$D = 5$	$D = 6$	$D = 5$	$D = 5$
$CX = 28$	$CX = 130$	$CX = 312$	$CX = 28$	$CX = 36$	$CX = 28$	$CX = 25.72$

The simple duration model (the first term) predicts that a 2 percent rise in interest rates will cause the bond's price to fall 9.25 percent. However, for large changes in yields, the duration model overpredicts the price fall. The duration model with the second-order convexity adjustment predicts a price fall of 8.69 percent; it adds back 0.56 percent due to the convexity effect. This is much closer to the true fall in the six-year, 8 percent coupon bond's price if we calculated this using 10 percent to discount the coupon and face value cash flows on the bond. The true value of the bond price fall is 8.71 percent. That is, using the convexity adjustment reduces the error between predicted value and true value to just a few basis points.<sup>26</sup>

In Table 3-15 we calculate various properties of convexity, where

- $N$  = Time to maturity
- $R$  = Yield to maturity
- $C$  = Annual coupon
- $D$  = Duration
- $CX$  = Convexity

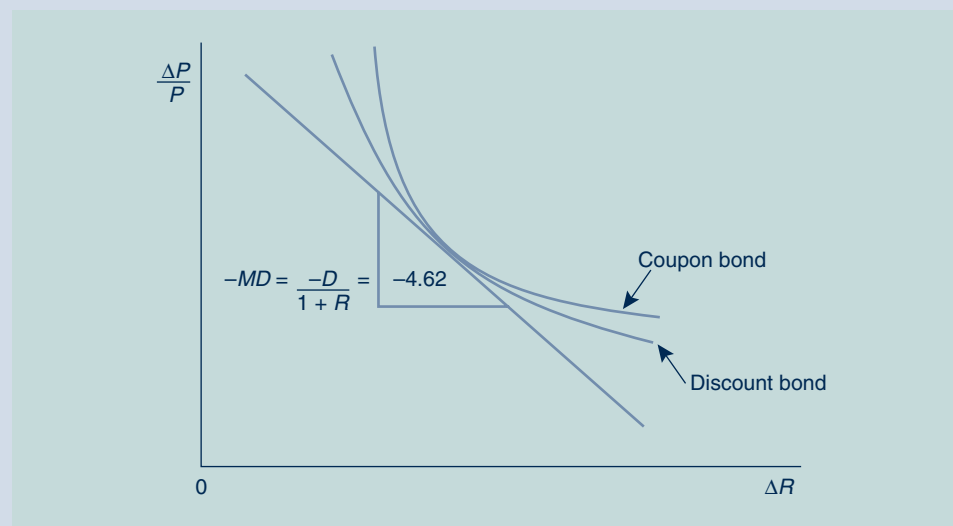
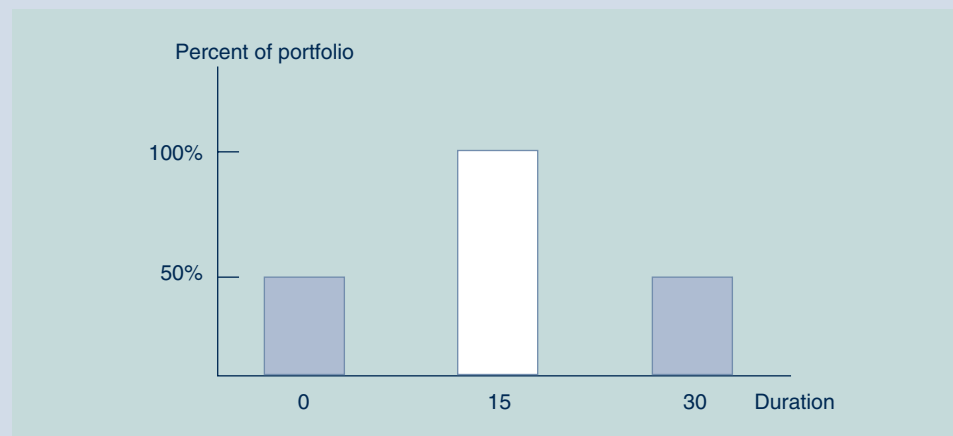
Part 1 of Table 3-15 shows that as the bond's maturity ( $N$ ) increases, so does its convexity ( $CX$ ). As a result, long-term bonds have more convexity—which is a desirable property—than do short-term bonds. This property is similar to that possessed by duration.<sup>27</sup>

Part 2 of Table 3-15 shows that coupon bonds of the same maturity ( $N$ ) have less convexity than do zero-coupon bonds. However, for coupon bonds and discount or zero-coupon bonds of the same duration, part 3 of the table shows that the coupon bond has more convexity. We depict the convexity of both in Figure 3-11.

Finally, before leaving convexity, we might look at one important use of the concept by managers of insurance companies, pension funds, and mutual funds. Remembering that convexity is a desirable form of interest rate risk insurance, FI managers could structure an asset portfolio to maximize its desirable effects. As an example, consider a pension fund manager with a 15-year payout horizon. To immunize the risk of interest rate changes, the manager purchases bonds with a 15-year duration. Consider two alternative strategies in achieve this:

26. It is possible to use the third moment of the Taylor series expansion to reduce this small error (8.71 percent versus 8.69 percent) even further. In practice, few people do this.

27. Note that the  $CX$  measure differs according to the level of interest rates. For example, we are measuring  $CX$  in Table 3-15 when yields are 8 percent. If yields were 12 percent, the  $CX$  number would change. This is intuitively reasonable, as the curvature of the price-yield curve differs at each point on the price-yield curve. Note that duration also changes with the level of interest rates.

**FIGURE 3–11 Convexity of a Coupon versus a Discount Bond with the Same Duration****FIGURE 3–12 Barbell Strategy**

Strategy 1: Invest 100 percent of resources in a 15-year deep-discount bond with an 8 percent yield.

Strategy 2: Invest 50 percent in the very short-term money market (federal funds) and 50 percent in 30-year deep-discount bonds with an 8 percent yield.

The duration ( $D$ ) and convexities ( $CX$ ) of these two asset portfolios are:

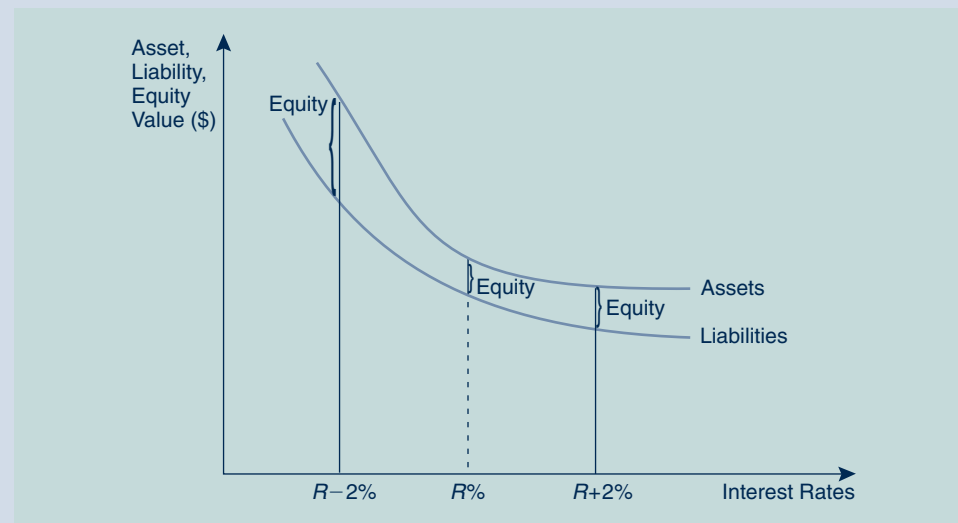
Strategy 1:  $D = 15$ ,  $CX = 206$

Strategy 2:<sup>28</sup>  $D = \frac{1}{2}(0) + \frac{1}{2}(30) = 15$ ,  $CX = \frac{1}{2}(0) + \frac{1}{2}(797) = 398.5$

Strategies 1 and 2 have the same durations, but strategy 2 has a greater convexity. Strategy 2 is often-called a barbell portfolio, as shown in Figure 3–12 by the shaded bars.<sup>29</sup> Strategy 1 is the unshaded bar. To the extent that the market does not price

28. The duration and convexity of one-day federal funds are approximately zero.

29. This is called a barbell because the weights are equally loaded at the extreme ends of the duration range or bar as in weight lifting.

**FIGURE 3–13 Assets Are More Convex Than Liabilities**

(or fully price) convexity, the barbell strategy dominates the direct duration matching strategy (number 1).<sup>30</sup>

More generally, an FI manager may seek to attain greater convexity in the asset portfolio than in the liability portfolio, as shown in Figure 3–13. As a result, both positive and negative shocks to interest rates would have beneficial effects on the FI's net worth.<sup>31</sup>

30. In a world in which convexity is priced, the long-term 30-year bond's price would rise to reflect the competition among buyers to include this more convex bond in their barbell asset portfolios. Thus, buying bond insurance—in the form of the barbell portfolio—would involve an additional cost to the FI manager. In addition, to be hedged in both a duration sense and a convexity sense, the manager should not choose the convexity of the asset portfolio without seeking to match it to the convexity of its liability portfolio. For further discussion of the convexity “trap” that results when an FI mismatches its asset and liability convexities, see J. H. Gilkeson and S. D. Smith, “The Convexity Trap: Pitfalls in Financing Mortgage Portfolios and Related Securities,” Federal Reserve Bank of Atlanta, *Economic Review*, November-December 1992, pp. 17-27.

31. Another strategy would be for the FI to issue callable bonds as liabilities. Callable bonds have limited upside capital gains because if rates fall to a low level, then the issuer calls the bond in early (and reissues new lower coupon bonds). The effect of limited upside potential for callable bond prices is that the price–yield curve for such bonds exhibits negative convexity. Thus, if asset investments have positive convexity and liabilities negative convexity, then yield shocks (whether positive or negative) are likely to produce net worth gains for the FI.